Sample Questions

These sample questions are drawn from major mathematics competitions and are similar in pattern to the upcoming Olympiad.

https://ivyleaguecenter.org/2017/09/29/35-sets-of-past-real-amc-12-tests-with-answer-keys/

https://www.askiitians.com/olympiads/inmo-question-papers-with-solutions/

https://igor-kortchemski.perso.math.cnrs.fr/olympiades/france_tst_2005.pdf

https://www.imo-official.org/problems/IMO2021SL.pdf

https://people.math.sc.edu/girardi/m142/integration/100problems.pdf

https://assets.ctfassets.net/xv1hr0zszyo1/1Bui41TER2cQ6aUqTIuHbO/cd46c2114f0b5083102d5cdfef6b0536/Integration_Bee_Questions_and_Answers.pdf

https://stevekifowit.com/bee/practice.pdf

https://cms.math.ca/wp-content/uploads/2024/03/CMO2024-problems.pdf

https://cms.math.ca/wp-content/uploads/2024/04/cmo2024-solutions-en.pdf

Integration Bee 2022

Round 1: Qualifiers

1. [2] $\displaystyle\int \frac{1}{1-e^{-2022x}}\,\mathrm{d}x$
2. [3] $\displaystyle\int \frac{\tan^3(1+\ln x)}{x}\,\mathrm{d}x$
3. [2] $\displaystyle\int \frac{1}{2-2x+x^2}\,\mathrm{d}x$
4. [2] $\displaystyle\int_{-1}^{0} \frac{15165}{2}x\sqrt{1+x}\,\mathrm{d}x$
5. [3] $\displaystyle\int_{0}^{1} (x-3^x)^2\,\mathrm{d}x$
6. [3] $\displaystyle\int \frac{x^{12}}{x^2+1}\,\mathrm{d}x$
7. [3] $\displaystyle\int \frac{x^2}{(x-1)(x^2+x+1)}\,\mathrm{d}x$
8. [2] $\displaystyle\int \frac{\sin 2x}{1+2\sin^2 x}\,\mathrm{d}x$
9. [4] $\displaystyle\int_{0}^{1} (2x-1)\frac{\sqrt{x}+\sqrt{1-x}}{\sqrt{x}-\sqrt{1-x}}\,\mathrm{d}x$
10. [3] $\displaystyle\int \frac{2\cos x-2\sin x}{2+\sin 2x}\,\mathrm{d}x$
11. [4] $\displaystyle\int_{0}^{e} x^{\frac{1}{\ln x}}\,\mathrm{d}x$
12. [4] $\displaystyle\int \sqrt{x\sqrt{x\sqrt{x\ldots}}}\,\mathrm{d}x$
13. [3] $\displaystyle\int_{1}^{2} x^{x^2+1}(2\ln x+1)\,\mathrm{d}x$
14. [4] $\displaystyle\int_{0}^{2022\pi} \sin(\sin x-x)\,\mathrm{d}x$
15. [3] $\displaystyle\int_{0}^{1} x\tan^{-1}\big((x-1)(x+1)\big)\,\mathrm{d}x$
16. [4] $\displaystyle\int \frac{1}{\sqrt{x\sqrt{x}-x^2}}\,\mathrm{d}x$
17. [7] $\displaystyle\int_{1}^{e^2} (\ln(x^{\ln x-6})+5)x^3\,\mathrm{d}x$
18. [5] $\displaystyle\int_{0}^{\pi/2} \frac{1}{\tan^{\sqrt{2022}}(x)+1}\,\mathrm{d}x$

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19. [6] $\displaystyle\int_0^1 \sqrt{x+\sqrt{x+\sqrt{x+\ldots}}}\,\mathrm{d}x$

20. [8] $\displaystyle\int_0^{\frac{\pi}{2}} \prod_{k=1}^{7} \cos kx\,\mathrm{d}x$

Round 2

Knockout Match 1

1. [1] $\displaystyle\int \sqrt{12-3x^2}\,\mathrm{d}x$

2. [2] $\displaystyle\int \frac{x}{1-5x^2}\sqrt{\frac{2}{1+5x^2}-1}\,\mathrm{d}x$

3. [2] $\displaystyle\int \frac{x^{1010}}{1-x^{2022}}\,\mathrm{d}x$

4. [2] $\displaystyle\int_{-\infty}^{0} x^{2022}e^{4x}\,\mathrm{d}x$

5. [3.5] $\displaystyle\int_0^1 2^{\lfloor \log_2 x \rfloor}\,\mathrm{d}x$

Knockout Match 2

1. [1] $\displaystyle\int \frac{8x^3}{1+x^8}\,\mathrm{d}x$

2. [2] $\displaystyle\int_{-\pi/4}^{\pi/4} \frac{1}{1-\sin x}\,\mathrm{d}x$

3. [2] $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos x(1+\tan^{-1}x)}{2-\cos^2 x}\,\mathrm{d}x$

4. [2] $\displaystyle\int_2^{\log_2 6} \frac{2^{2x+1}-9\times 2^x-6}{4^x-2^x-6}\,\mathrm{d}x$

5. [3.5] Define $f(x)$ as the $\lfloor x \rfloor$th decimal place of x. Find $\displaystyle\int_1^{2022} f(x)\,\mathrm{d}x$.

Knockout Match 3

1. [2] $\displaystyle\int_0^{\infty} \frac{1}{(x+\frac{1}{x})^2}\,\mathrm{d}x$

2. [2.5] $\displaystyle\int_{\frac{1}{2}}^{1} \sum_{k=0}^{\infty} \left(x^k\left((k+1)\ln x + x^{k^2+k}\right)\right)\mathrm{d}x$

3. [3.5] $\displaystyle\int_0^{\infty} \frac{1}{(1+x^{2022})(1+x^2)}\,\mathrm{d}x$

4. [3.5] $\displaystyle\int_0^{10} \lim_{s\to x^+}\left(\lim_{t\to\infty} t^{-\frac{|\sin s|+\sin s}{\sin s}}\right)\mathrm{d}x$

5. [4] $\displaystyle\int_0^{\infty} \sum_{k=\lfloor x \rfloor}^{\infty} \frac{2^{-k}}{k+1}\,\mathrm{d}x$

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Extra Questions

1. [1] $\displaystyle\int \frac{\sin^3 x}{\sqrt{\cos x}}\,\mathrm{d}x$

2. [1] $\displaystyle\int_0^1 \frac{(\sin^{-1} x)^2}{\sqrt{1-x^2}}\,\mathrm{d}x$

3. [2] $\displaystyle\int \ln\left(1+\sqrt{x}\right)\mathrm{d}x$

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Answers

Round 1

1. [2] $\frac{1}{2022}\ln|e^{2000x}-1|+C$
2. [3] $\frac{1}{2}\tan^2(1+\ln x)+\ln|\cos(1+\ln x)|+C$
3. [2] $\tan^{-1}(x-1)+C$
4. [2] -2022
5. [3] $\frac{1}{3}-\frac{2}{\ln 3}+\frac{4}{(\ln 3)^2}$
6. [3] $\frac{x^{11}}{11}-\frac{x^9}{9}+\frac{x^7}{7}-\frac{x^5}{5}+\frac{x^3}{3}-x+\tan^{-1}x+C$
7. [3] $\frac{1}{3}\ln(x^3-1)+C$
8. [2] $\frac{1}{2}\ln(2-\cos 2x)+C$
9. [4] $1+\frac{\pi}{4}$
10. [3] $2\tan^{-1}(\sin x+\cos x)+C$
11. [4] e^2
12. [4] $\frac{x^2}{2}+C$
13. [3] 15
14. [4] 0
15. [3] $-\frac{\pi}{8}+\frac{1}{4}\ln 2$
16. [4] $4\sin^{-1}(x^{1/4})+C$
17. [7] $-\frac{53}{32}-\frac{19}{32}e^8$
18. [5] $\frac{\pi}{4}$
19. [6] $\frac{5}{12}(1+\sqrt{5})$
20. [8] $\frac{\pi}{32}$

Knockout Match 1

1. [1] $\frac{\sqrt{3}}{2}(x\sqrt{4-x^2}+4\sin^{-1}(\frac{x}{2}))+C$
2. [2] $\frac{1}{10}\sin^{-1}(5x^2)+C$
3. [2] $\frac{1}{2022}\ln|\frac{1+x^{1011}}{1-x^{1011}}|+C$
4. [2] $\frac{2022!}{2^{4046}}$
5. [3.5] $\frac{1}{3}$

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Knockout Match 2

1. [1] $2\tan^{-1}(x^4) + C$
2. [2] 2
3. [2] $\frac{\pi}{2}$
4. [2] $3 - 2\log_2 3$
5. [3.5] 9094.5

Finals

1. [2] $\frac{\pi}{4}$
2. [2.5] $3\ln 2$
3. [3.5] $\frac{\pi}{4}$
4. [3.5] $10 - 2\pi$
5. [4] 2

Extra Questions

1. [1] $\frac{2}{5}\cos^{\frac{5}{2}} x - 2\cos^{\frac{1}{2}} x + C$
2. [1] $\frac{\pi^3}{24}$
3. [2] $(x-1)\ln(1+\sqrt{x}) + \sqrt{x} - \frac{x}{2} + C$

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Sample Problems Touching All Parts of the Curriculum for the Lagos-Virginia Cross Continental Mathematical Olympiad

Exam Start Date is March 16, 2024

AMC 12A 2021

1 What is the value of

$$2^{1+2+3} - (2^1 + 2^2 + 2^3)?$$

(A) 0 **(B)** 50 **(C)** 52 **(D)** 54 **(E)** 57

2 Under what conditions is $\sqrt{a^2 + b^2} = a + b$ true, where a and b are real numbers?

(A) It is never true. **(B)** It is true if and only if $ab = 0$. **(C)** It is true if and only if $a + b \geq 0$. **(D)** It is true if and only if $ab = 0$ and $a + b \geq 0$. **(E)** It is always true.

3 The sum of two natural numbers is $17,402$. One of the two numbers is divisible by 10. If the units digit of that number is erased, the other number is obtained. What is the difference of these two numbers?

(A) $10,272$ **(B)** $11,700$ **(C)** $13,362$ **(D)** $14,238$ **(E)** $15,426$

4 Tom has a collection of 13 snakes, 4 of which are purple and 5 of which are happy. He observes that • all of his happy snakes can add • none of his purple snakes can subtract, and • all of his snakes that cant subtract also cant add

Which of these conclusions can be drawn about Toms snakes?
(A) Purple snakes can add.
(B) Purple snakes are happy.
(C) Snakes that can add are purple.
(D) Happy snakes are not purple.
(E) Happy snakes can't subtract.

5 When a student multiplied the number 66 by the repeating decimal,

$$1.\underline{abab} = 1.\overline{ab},$$

where a and b are digits, he did not notice the notation and just multiplied 66 times $1.\underline{ab}$. Later he found that his answer is 0.5 less than the correct answer. What is the 2- digit integer $\underline{ab}$?

(A) 15 **(B)** 30 **(C)** 45 **(D)** 60 **(E)** 75

6 A deck of cards has only red cards and black cards. The probability of a randomly chosen card being red is $\frac{1}{3}$. When 4 black cards are added to the deck, the probability of choosing red becomes $\frac{1}{4}$. How many cards were in the deck originally.

(A) 6 **(B)** 9 **(C)** 12 **(D)** 15 **(E)** 18

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7 What is the least possible value of $(xy-1)^2+(x+y)^2$ for real numbers x and y?

(A) 0 **(B)** $\frac{1}{4}$ **(C)** $\frac{1}{2}$ **(D)** 1 **(E)** 2

---

8 A sequence of numbers is defined by $D_0=0, D_1=0, D_2=1$ and $D_n=D_{n-1}+D_{n-3}$ for $n \geq 3$. What are the parities (evenness or oddness) of the triple of numbers $(D_{2021}, D_{2022}, D_{2023})$, where E denotes even and O denotes odd?

(A) (O,E,O) **(B)** (E,E,O) **(C)** (E,O,E) **(D)** (O,O,E) **(E)** (O,O,O)

---

9 Which of the following is equivalent to

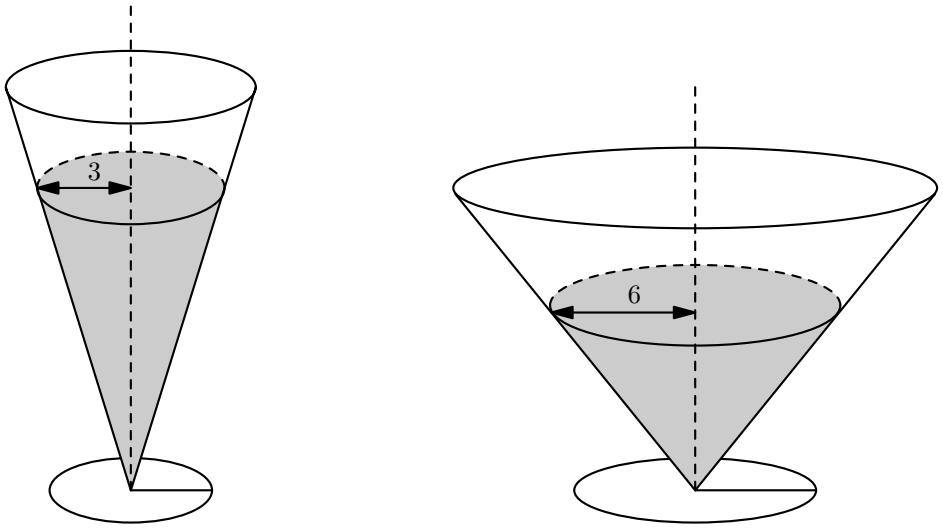
$$(2+3)(2^2+3^2)(2^4+3^4)(2^8+3^8)(2^{16}+3^{16})(2^{32}+3^{32})(2^{64}+3^{64})?$$

(A) $3^{127}+2^{127}$ **(B)** $3^{127}+2^{127}+2\cdot 3^{63}+3\cdot 2^{63}$ **(C)** $3^{128}-2^{128}$ **(D)** $3^{128}+2^{128}$ **(E)** 5^{127}

---

10 Two right circular cones with vertices facing down as shown in the figure below contain the same amount of liquid. The radii of the tops of the liquid surfaces are 3 cm and 6 cm. Into each cone is dropped a spherical marble of radius 1 cm, which sinks to the bottom and is completely submerged without spilling any liquid. What is the ratio of the rise of the liquid level in the narrow cone to the rise of the liquid level in the wide cone?

(A) $1:1$ **(B)** $47:43$ **(C)** $2:1$ **(D)** $40:13$ **(E)** $4:1$



11 A laser is placed at the point (3,5). The laser bean travels in a straight line. Larry wants the beam to hit and bounce off the y-axis, then hit and bounce off the x-axis, then hit the point $(7, 5)$. What is the total distance the beam will travel along this path?

(A) $2\sqrt{10}$ **(B)** $5\sqrt{2}$ **(C)** $10\sqrt{2}$ **(D)** $15\sqrt{2}$ **(E)** $10\sqrt{5}$

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12 All the roots of polynomial $z^6 - 10z^5 + Az^4 + Bz^3 + Cz^2 + Dz + 16$ are positive integers. What is the value of B?

(A) -88 **(B)** -80 **(C)** -64 **(D)** -41 **(E)** -40

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13 Of the following complex numbers z, which one has the property that z^5 has the greatest real part?

(A) -2 **(B)** $-\sqrt{3} + i$ **(C)** $-\sqrt{2} + \sqrt{2}i$ **(D)** $-1 + \sqrt{3}i$ **(E)** $2i$

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14 What is the value of

$$\left(\sum_{k=1}^{20} \log_{5^k} 3^{k^2}\right) \cdot \left(\sum_{k=1}^{100} \log_{9^k} 25^k\right)?$$

(A) 21 **(B)** $100 \log_5 3$ **(C)** $200 \log_3 5$ **(D)** $2,200$ **(E)** $21,000$

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15 A choir director must select a group of singers from among his 6 tenors and 8 basses. The only requirements are that the difference between the number of tenors and basses must be a multiple of 4, and the group must have at least one singer. Let N be the number of groups that can be selected. What is the remainder when N is divided by 100?

(A) 47 **(B)** 48 **(C)** 83 **(D)** 95 **(E)** 96

---

16 In the following list of numbers, the integer n appears n times in the list for $1 \leq n \leq 200$.

$$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, , 200, 200, , 200$$

What is the median of the numbers in this list?

(A) 100.5 **(B)** 134 **(C)** 142 **(D)** 150.5 **(E)** 167

---

17 Trapezoid $ABCD$ has $\overline{AB} \parallel \overline{CD}$, $BC = CD = 43$, and $\overline{AD} \perp \overline{BD}$. Let O be the intersection of the diagonals $\overline{AC}$ and $\overline{BD}$, and let P be the midpoint of $\overline{BD}$. GIven that $OP = 11$, the length AD can be written in the form $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. What is $m + n$?

(A) 65 **(B)** 132 **(C)** 157 **(D)** 194 **(E)** 215

18 Let f be a function defined on the set of positive rational numbers with the property that $f(a \cdot b) = f(a) + f(b)$ for all positive rational numbers a and b. Suppose that f also has the property that $f(p) = p$ for every prime number p. For which of the following numbers x is $f(x) < 0$?

(A) $\frac{17}{32}$ **(B)** $\frac{11}{16}$ **(C)** $\frac{7}{9}$ **(D)** $\frac{7}{6}$ **(E)** $\frac{25}{11}$

---

19 How many solutions does the equation $\sin\left(\frac{\pi}{2}\cos x\right) = \cos\left(\frac{\pi}{2}\sin x\right)$ have in the closed interval $[0, \pi]$?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

---

20 Suppose that on a parabola with vertex V and a focus F there exists a point A such that $AF = 20$ and $AV = 21$. What is the sum of all possible values of the length FV?

(A) 13 **(B)** $\frac{40}{3}$ **(C)** $\frac{41}{3}$ **(D)** 14 **(E)** $\frac{43}{3}$

---

21 The five solutions to the equation

$$(z-1)(z^2+2z+4)(z^2+4z+6) = 0$$

may be written in the form $x_k + y_k i$ for $1 \leq k \leq 5$, where x_k and y_k are real. Let $\mathcal{E}$ be the unique ellipse that passes through the points $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$, and (x_5, y_5). The eccentricity of $\mathcal{E}$ can be written in the form $\sqrt{\frac{m}{n}}$ where m and n are relatively prime positive integers. What is $m + n$? (Recall that the *eccentricity* of an ellipse $\mathcal{E}$ is the ratio $\frac{c}{a}$, where $2a$ is the length of the major axis of E and $2c$ is the is the distence between its two foci.)

(A) 7 **(B)** 9 **(C)** 11 **(D)** 13 **(E)** 15

---

22 Suppose that the roots of the polynomial $P(x) = x^3 + ax^2 + bx + c$ are $\cos\frac{2\pi}{7}, \cos\frac{4\pi}{7}$, and $\cos\frac{6\pi}{7}$, where angles are in radians. What is abc?

(A) $-\frac{3}{49}$ **(B)** $-\frac{1}{28}$ **(C)** $\frac{\sqrt[3]{7}}{64}$ **(D)** $\frac{1}{32}$ **(E)** $\frac{1}{28}$

---

23 Frieda the frog begins a sequence of hops on a 3×3 grid of squares, moving one square on each hop and choosing at random the direction of each hop up, down, left, or right. She does not hop diagonally. When the direction of a hop would take Frieda off the grid, she "wraps around" and jumps to the opposite edge. For example if Frieda begins in the center square and makes two hops "up", the first hop would place her in the top row middle square, and the second hop would cause Frieda to jump to the opposite edge, landing in the bottom row middle square. Suppose Frieda starts from the center square, makes at most four hops at random, and stops hopping if she lands on a corner square. What is the probability that she reaches a corner square on one of the four hops?

(A) $\frac{9}{16}$ **(B)** $\frac{5}{8}$ **(C)** $\frac{3}{4}$ **(D)** $\frac{25}{32}$ **(E)** $\frac{13}{16}$

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24 Semicircle Γ has diameter $\overline{AB}$ of length 14. Circle Ω lies tangent to $\overline{AB}$ at a point P and intersects Γ at points Q and R. If $QR = 3\sqrt{3}$ and $\angle QPR = 60°$, then the area of $\triangle PQR$ is $\frac{a\sqrt{b}}{c}$, where a and c are relatively prime positive integers, and b is a positive integer not divisible by the square of any prime. What is $a + b + c$?

(A) 110 **(B)** 114 **(C)** 118 **(D)** 122 **(E)** 126

---

25 Let $d(n)$ denote the number of positive integers that divide n, including 1 and n. For example, $d(1) = 1, d(2) = 2$, and $d(12) = 6$. (This function is known as the *divisor function*.) Let

$$f(n) = \frac{d(n)}{\sqrt[3]{n}}.$$

There is a unique positive integer N such that $f(N) > f(n)$ for all positive integers $n \neq N$. What is the sum of the digits of N?

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

AMC 12B 2021

1 How many integer values satisfy $|x| < 3\pi$?

(A) 9 **(B)** 10 **(C)** 18 **(D)** 19 **(E)** 20

2 At a math contest, 57 students are wearing blue shirts, and another 75 students are wearing yellow shirts. The 132 students are assigned into 66 points. In exactly 23 of these pairs, both students are wearing blue shirts. In how many pairs are both studets wearing yellow shirts?

(A) 23 **(B)** 32 **(C)** 37 **(D)** 41 **(E)** 64

3 Suppose

$$2+\frac{1}{1+\frac{1}{2+\frac{2}{3+x}}}=\frac{144}{53}.$$

What is the value of x?

(A) $\frac{3}{4}$ **(B)** $\frac{7}{8}$ **(C)** $\frac{14}{15}$ **(D)** $\frac{37}{38}$ **(E)** $\frac{52}{53}$

4 Ms. Blackwell gives an exam to two classes. The mean of the scores of the students in the morning class is 84, and the afternoon classs mean score is 70. The ratio of the number of students in the morning class to the number of students in the afternoon class is $\frac{3}{4}$. What is the mean of the scores of all the students?

(A) 74 **(B)** 75 **(C)** 76 **(D)** 77 **(E)** 78

5 The point $P(a,b)$ in the xy-plane is first rotated counterclockwise by 90° around the point $(1,5)$ and then reflected about the line $y=-x$. The image of P after these two transformations is at $(-6,3)$. What is $b-a$?

(A) 1 **(B)** 3 **(C)** 5 **(D)** 7 **(E)** 9

6 An inverted cone with base radius 12 cm and height 18 cm is full of water. The water is poured into a tall cylinder whose horizontal base has a radius of 24 cm. What is the height in centimeters of the water in the cylinder?

(A) 1.5 **(B)** 3 **(C)** 4 **(D)** 4.5 **(E)** 6

7 Let $N=34\cdot 34\cdot 63\cdot 270$. What is the ratio of the sum of the odd divisors of N to the sum of the even divisors of N?

(A) $1:16$ **(B)** $1:15$ **(C)** $1:14$ **(D)** $1:8$ **(E)** $1:3$

8 Three equally spaced parallel lines intersect a circle, creating three chords of lengths $38, 38,$ and 34. What is the distance between two adjacent parallel lines?

(A) $5\frac{1}{2}$ **(B)** 6 **(C)** $6\frac{1}{2}$ **(D)** 7 **(E)** $7\frac{1}{2}$

---

9 What is the value of

$$\frac{\log_2 80}{\log_{40} 2} - \frac{\log_2 160}{\log_{20} 2}?$$

(A) 0 **(B)** 1 **(C)** $\frac{5}{4}$ **(D)** 2 **(E)** $\log_2 5$

---

10 Two distinct numbers are selected from the set $\{1, 2, 3, 4, \ldots, 36, 37\}$ so that the sum of the remaining 35 numbers is the product of these two numbers. What is the difference of these two numbers?

(A) 5 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 10

---

11 Triangle ABC has $AB = 13, BC = 14$ and $AC = 15$. Let P be the point on $\overline{AC}$ such that $PC = 10$. There are exactly two points D and E on line BP such that quadrilaterals $ABCD$ and $ABCE$ are trapezoids. What is the distance DE?

(A) $\frac{42}{5}$ **(B)** $6\sqrt{2}$ **(C)** $\frac{84}{5}$ **(D)** $12\sqrt{2}$ **(E)** 18

---

12 Suppose that S is a finite set of positive integers. If the greatest integer in S is removed from S, then the average value (arithmetic mean) of the integers remaining is 32. If the least integer is S is *also* removed, then the average value of the integers remaining is 35. If the greatest integer is then returned to the set, the average value of the integers rises to 40. The greatest integer in the original set S is 72 greater than the least integer in S. What is the average value of all the integers in the set S?

(A) 36.2 **(B)** 36.4 **(C)** 36.6 **(D)** 36.8 **(E)** 37

---

13 How many values of θ in the interval $0 < \theta \leq 2\pi$ satisfy

$$1 - 3\sin\theta + 5\cos 3\theta = 0?$$

(A) 2 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 8

---

14 Let $ABCD$ be a rectangle and let $\overline{DM}$ be a segment perpendicular to the plane of $ABCD$. Suppose that $\overline{DM}$ has integer length, and the lengths of $\overline{MA}, \overline{MC}$, and $\overline{MB}$ are consecutive odd positive integers (in this order). What is the volume of pyramid $MABCD$?

(A) $24\sqrt{5}$ **(B)** 60 **(C)** $28\sqrt{5}$ **(D)** 66 **(E)** $8\sqrt{70}$

---

15 The figure below is constructed from 11 line segments, each of which has length 2. The area of pentagon $ABCDE$ can be written as $\sqrt{m} + \sqrt{n}$, where m and n are positive integers. What is $m + n$?

(A) 20 **(B)** 21 **(C)** 22 **(D)** 23 **(E)** 24

16 Let $g(x)$ be a polynomial with leading coefficient 1, whose three roots are the reciprocals of the three roots of $f(x) = x^3 + ax^2 + bx + c$, where $1 < a < b < c$. What is $g(1)$ in terms of a, b, and c?

(A) $\frac{1+a+b+c}{c}$ **(B)** $1 + a + b + c$ **(C)** $\frac{1+a+b+c}{c^2}$ **(D)** $\frac{a+b+c}{c^2}$ **(E)** $\frac{1+a+b+c}{a+b+c}$

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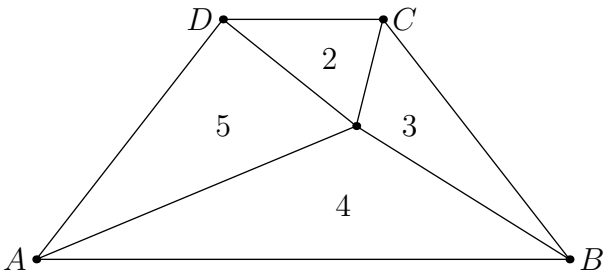
17 Let $ABCD$ be an isoceles trapezoid having parallel bases $\overline{AB}$ and $\overline{CD}$ with $AB > CD$. Line segments from a point inside $ABCD$ to the vertices divide the trapezoid into four triangles whose areas are $2, 3, 4$, and 5 starting with the triangle with base $\overline{CD}$ and moving clockwise as shown in the diagram below. What is the ratio $\frac{AB}{CD}$?



(A) 3 **(B)** $2 + \sqrt{2}$ **(C)** $1 + \sqrt{6}$ **(D)** $2\sqrt{3}$ **(E)** $3\sqrt{2}$

---

18 Let z be a complex number satisfying $12|z|^2 = 2|z + 2|^2 + |z^2 + 1|^2 + 31$. What is the value of $z + \frac{6}{z}$?

(A) -2 **(B)** -1 **(C)** $\frac{1}{2}$ **(D)** 1 **(E)** 4

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19 Two fair dice, each with at least 6 faces, are rolled. On each face of each die is printed a distinct integer from 1 to the number of faces on that die, inclusive. The probability of rolling a sum of 7 is $\frac{3}{4}$ of the probability of rolling a sum of 10 and the probability of rolling a sum of 12 is $\frac{1}{12}$. What is the least possible number of faces on the two dice combined?

(A) 16 **(B)** 17 **(C)** 18 **(D)** 19 **(E)** 20

---

20 Let $Q(z)$ and $R(z)$ be the unique polynomials such that

$$z^{2021} + 1 = (z^2 + z + 1)Q(z) + R(z)$$

and the degree of R is less than 2. What is $R(z)$?

(A) $-z$ **(B)** -1 **(C)** 2021 **(D)** $z + 1$ **(E)** $2z + 1$

21 Let S be the sum of all positive real numbers x for which

$$x^{2^{\sqrt{2}}} = \sqrt{2}^{2^x}.$$

Which of the following statements is true?

(A) $S < \sqrt{2}$ **(B)** $S = \sqrt{2}$ **(C)** $\sqrt{2} < S < 2$ **(D)** $2 \leq S < 6$ **(E)** $S \geq 6$

22 Arjun and Beth play a game in which they take turns removing one brick or two adjacent bricks from one "wall" among a set of several walls of bricks, with gaps possibly creating new walls. The walls are one brick tall. For example, a set of walls of sizes 4 and 2 can be changed into any of the following by one move: $(3, 2), (2, 1, 2), (4), (4, 1), (2, 2),$ or $(1, 1, 2)$.
Arjun plays first, and the player who removes the last
brick wins. For which starting configuration is there a strategy that guarantees a win for Beth?

(A) $(6, 1, 1)$ **(B)** $(6, 2, 1)$ **(C)** $(6, 2, 2)$ **(D)** $(6, 3, 1)$ **(E)** $(6, 3, 2)$

23 Three balls are randomly and independently tossed into bins numbered with the positive integers so that for each ball, the probability it is tossed into bin i is 2^{-i} for $i = 1, 2, 3, \ldots$. More than one ball is allowed in each bin. The probability that the balls end up evenly spaced in distinct bins is $\frac{p}{q}$, where p and q are relatively prime positive integers. (For example, the balls are evenly spaced if they are tossed into bins 3, 17, and 10.) What is $p + q$?

(A) 55 **(B)** 56 **(C)** 57 **(D)** 58 **(E)** 59

24 Let $ABCD$ be a parallelogram with area 15. Points P and Q are the projections of A and C, respectively, onto the line BD; and points R and S are the projections of B and D, respectively, onto the line AC. See the figure, which also shows the relative locations of these points.

Suppose $PQ = 6$ and $RS = 8$, and let d denote the

length of $\overline{BD}$, the longer diagonal of $ABCD$. Then d^2 can be written in the form $m + n\sqrt{p}$, where m, n, and p are positive integers and p is not divisible by the square of any prime. What is $m + n + p$?

(A) 81 **(B)** 89 **(C)** 97 **(D)** 105 **(E)** 113

25 Let S be the set of lattice points in the coordinate plane, both of whose coordinates are integers between 1 and 30, inclusive. Exactly 300 points in S lie on or below a line with equation $y = mx$. The possible values of m lie in an interval of length $\frac{a}{b}$, where a and b are relatively prime positive integers. What is $a + b$?

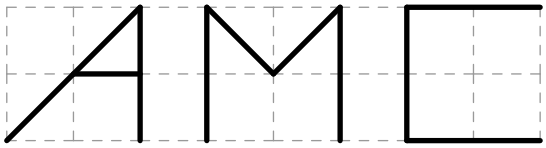
(A) 31 **(B)** 47 **(C)** 62 **(D)** 72 **(E)** 85

AMC12A 2020

1 Carlos took 70% of a whole pie. Maria took one third of the remainder. What portion of the whole pie was left?

(A) 10% **(B)** 15% **(C)** 20% **(D)** 30% **(E)** 35%

2 The acronym AMC is shown in the rectangular grid below with grid lines spaced 1 unit apart. In units, what is the sum of the lengths of the line segments that form the acronym AMC?



(A) 17 **(B)** $15+2\sqrt{2}$ **(C)** $13+4\sqrt{2}$ **(D)** $11+6\sqrt{2}$ **(E)** 21

3 A driver travels for 2 hours at 60 miles per hour, during which her car gets 30 miles per gallon of gasoline. She is paid $\$0.50$ per mile, and her only expense is gasoline at $\$2.00$ per gallon. What is her net rate of pay, in dollars per hour, after this expense?

(A) 20 **(B)** 22 **(C)** 24 **(D)** 25 **(E)** 26

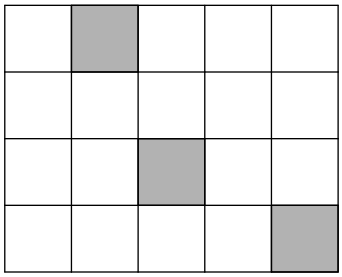
4 How many 4-digit positive integers (that is, integers between 1000 and 9999, inclusive) having only even digits are divisible by 5?

(A) 80 **(B)** 100 **(C)** 125 **(D)** 200 **(E)** 500

5 The 25 integers from -10 to 14, inclusive, can be arranged to form a 5-by-5 square in which the sum of the numbers in each row, the sum of the numbers in each column, and the sum of the numbers along each of the main diagonals are all the same. What is the value of this common sum?

(A) 2 **(B)** 5 **(C)** 10 **(D)** 25 **(E)** 50

6 In the plane figure shown below, 3 of the unit squares have been shaded. What is the least number of additional unit squares that must be shaded so that the resulting figure has two lines of symmetry?



(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

7 Seven cubes, whose volumes are $1, 8, 27, 64, 125, 216$, and 343 cubic units, are stacked vertically to form a tower in which the volumes of the cubes decrease from bottom to top. Except for the bottom cube, the bottom face of each cube lies completely on top of the cube below it. What is the total surface area of the tower (including the bottom) in square units?

(A) 644 **(B)** 658 **(C)** 664 **(D)** 720 **(E)** 749

---

8 What is the median of the following list of 4040 numbers?

$$1, 2, 3, ..., 2020, 1^2, 2^2, 3^2, ..., 2020^2$$

(A) 1974.5 **(B)** 1975.5 **(C)** 1976.5 **(D)** 1977.5 **(E)** 1978.5

---

9 How many solutions does the equation $\tan(2x) = \cos(\frac{x}{2})$ have on the interval $[0, 2\pi]$?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

---

10 There is a unique positive integer n such that

$$\log_2(\log_{16} n) = \log_4(\log_4 n).$$

What is the sum of the digits of n?

(A) 4 **(B)** 7 **(C)** 8 **(D)** 11 **(E)** 13

---

11 A frog sitting at the point $(1, 2)$ begins a sequence of jumps, where each jump is parallel to one of the coordinate axes and has length 1, and the direction of each jump (up, down, right, or left) is chosen independently at random. The sequence ends when the frog reaches a side of the square with vertices $(0, 0), (0, 4), (4, 4)$, and $(4, 0)$. What is the probability that the sequence of jumps ends on a vertical side of the square?

(A) $\frac{1}{2}$ **(B)** $\frac{5}{8}$ **(C)** $\frac{2}{3}$ **(D)** $\frac{3}{4}$ **(E)** $\frac{7}{8}$

---

12 Line ℓ in the coordinate plane has the equation $3x - 5y + 40 = 0$. This line is rotated $45°$ counterclockwise about the point $(20, 20)$ to obtain line k. What is the x-coordinate of the x-intercept of line k?

(A) 10 **(B)** 15 **(C)** 20 **(D)** 25 **(E)** 30

---

13 There are integers a, b, and c, each greater than 1, such that

$$\sqrt[a]{N\sqrt[b]{N\sqrt[c]{N}}} = \sqrt[36]{N^{25}}$$

for all $N > 1$. What is b?

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

14 Regular octagon $ABCDEFGH$ has area n. Let m be the area of quadrilateral $ACEG$. What is $\frac{m}{n}$?

(A) $\frac{\sqrt{2}}{4}$ **(B)** $\frac{\sqrt{2}}{2}$ **(C)** $\frac{3}{4}$ **(D)** $\frac{3\sqrt{2}}{5}$ **(E)** $\frac{2\sqrt{2}}{3}$

---

15 In the complex plane, let A be the set of solutions to $z^3-8=0$ and let B be the set of solutions to $z^3-8z^2-8z+64=0$. What is the greatest distance between a point of A and a point of B?

(A) $2\sqrt{3}$ **(B)** 6 **(C)** 9 **(D)** $2\sqrt{21}$ **(E)** $9+\sqrt{3}$

---

16 A point is chosen at random within the square in the coordinate plane whose vertices are $(0,0), (2020,0), (2020,2020)$, and $(0,2020)$. The probability that the point is within d units of a lattice point is $\frac{1}{2}$. (A point (x,y) is a lattice point if x and y are both integers.) What is d to the nearest tenth?

(A) 0.3 **(B)** 0.4 **(C)** 0.5 **(D)** 0.6 **(E)** 0.7

---

17 The vertices of a quadrilateral lie on the graph of $y=\ln x$, and the x-coordinates of these vertices are consecutive positive integers. The area of the quadrilateral is $\ln\frac{91}{90}$. What is the x-coordinate of the leftmost vertex?

(A) 6 **(B)** 7 **(C)** 10 **(D)** 12 **(E)** 13

---

18 Quadrilateral $ABCD$ satisfies $\angle ABC=\angle ACD=90^\circ$, $AC=20$, and $CD=30$. Diagonals $\overline{AC}$ and $\overline{BD}$ intersect at point E, and $AE=5$. What is the area of quadrilateral $ABCD$?

(A) 330 **(B)** 340 **(C)** 350 **(D)** 360 **(E)** 370

---

19 There exists a unique strictly increasing sequence of nonnegative integers $a_1<a_2<\ <a_k$ such that

$$\frac{2^{289}+1}{2^{17}+1}=2^{a_1}+2^{a_2}+\ +2^{a_k}.$$

What is k?

(A) 117 **(B)** 136 **(C)** 137 **(D)** 273 **(E)** 306

---

20 Let T be the triangle in the coordinate plane with vertices $(0,0)$, $(4,0)$, and $(0,3)$. Consider the following five isometries (rigid transformations) of the plane: rotations of 90°, 180°, and 270° counterclockwise around the origin, reflection across the x-axis, and reflection across the y-axis. How many of the 125 sequences of three of these transformations (not necessarily distinct) will return T to its original position? (For example, a 180° rotation, followed by a reflection across the x-axis, followed by a reflection across the y-axis will return T to its original position, but a 90° rotation, followed by a reflection across the x-axis, followed by another reflection across the x-axis will not return T to its original position.)

(A) 12 **(B)** 15 **(C)** 17 **(D)** 20 **(E)** 25

21 How many positive integers n are there such that n is a multiple of 5, and the least common multiple of $5!$ and n equals 5 times the greatest common divisor of $10!$ and n?

(A) 12 **(B)** 24 **(C)** 36 **(D)** 48 **(E)** 72

---

22 Let (a_n) and (b_n) be the sequences of real numbers such that

$$(2+i)^n = a_n + b_n i$$

for all integers $n \geq 0$, where $i = \sqrt{-1}$. What is

$$\sum_{n=0}^{\infty} \frac{a_n b_n}{7^n} \, ?$$

(A) $\frac{3}{8}$ **(B)** $\frac{7}{16}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{9}{16}$ **(E)** $\frac{4}{7}$

---

23 Jason rolls three fair standard six-sided dice. Then he looks at the rolls and chooses a subset of the dice (possibly empty, possibly all three dice) to reroll. After rerolling, he wins if and only if the sum of the numbers face up on the three dice is exactly 7. Jason always plays to optimize his chances of winning. What is the probability that he chooses to reroll exactly two of the dice?

(A) $\frac{7}{36}$ **(B)** $\frac{5}{24}$ **(C)** $\frac{2}{9}$ **(D)** $\frac{17}{72}$ **(E)** $\frac{1}{4}$

---

24 Suppose that $\triangle ABC$ is an equilateral triangle of side length s, with the property that there is a unique point P inside the triangle such that $AP = 1$, $BP = \sqrt{3}$, and $CP = 2$. What is s?

(A) $1+\sqrt{2}$ **(B)** $\sqrt{7}$ **(C)** $\frac{8}{3}$ **(D)** $\sqrt{5+\sqrt{5}}$ **(E)** $2\sqrt{2}$

---

25 The number $a = \frac{p}{q}$, where p and q are relatively prime positive integers, has the property that the sum of all real numbers x satisfying

$$\lfloor x \rfloor \cdot \{x\} = a \cdot x^2$$

is 420, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x and $\{x\} = x - \lfloor x \rfloor$ denotes the fractional part of x. What is $p+q$?

(A) 245 **(B)** 593 **(C)** 929 **(D)** 1331 **(E)** 1332

AMC 12B 2019

1 Alicia had two containers. The first was $\frac{5}{6}$ full of water and the second was empty. She poured all the water from the first container into the second container, at which point the second container was $\frac{3}{4}$ full of water. What is the ratio of the volume of the first container to the volume of the second container?

(A) $\frac{5}{8}$ **(B)** $\frac{4}{5}$ **(C)** $\frac{7}{8}$ **(D)** $\frac{9}{10}$ **(E)** $\frac{11}{12}$

---

2 Consider the statement, "If n is not prime, then $n-2$ is prime." Which of the following values of n is a counterexample to this statement?

(A) 11 **(B)** 15 **(C)** 19 **(D)** 21 **(E)** 27

---

3 Which one of the following rigid transformations (isometries) maps the line segment $\overline{AB}$ onto the line segment $\overline{A'B'}$ so that the image of $A(-2,1)$ is $A'(2,-1)$ and the image of $B(-1,4)$ is $B'(1,-4)$?

(A) reflection in the y-axis **(B)** counterclockwise rotation around the origin by 90° **(C)** translation by 3 units to the right and 5 units down **(D)** reflection in the x-axis **(E)** clockwise rotation about the origin by 180°

---

4 A positive integer n satisfies the equation $(n+1)!+(n+2)!=n!\cdot 440$. What is the sum of the digits of n?

(A) 2 **(B)** 5 **(C)** 10 **(D)** 12 **(E)** 15

---

5 Each piece of candy in a store costs a whole number of cents. Casper has exactly enough money to buy either 12 pieces of red candy, 14 pieces of green candy, 15 pieces of blue candy, or n pieces of purple candy. A piece of purple candy costs 20 cents. What is the smallest possible value of n?

(A) 18 **(B)** 21 **(C)** 24 **(D)** 25 **(E)** 28

6 In a given plane, points A and B are 10 units apart. How many points C are there in the plane such that the perimeter of $\triangle ABC$ is 50 units and the area of $\triangle ABC$ is 100 square units?

(A) 0 **(B)** 2 **(C)** 4 **(D)** 8 **(E)** infinitely many

7 What is the sum of all real numbers x for which the median of the numbers $4, 6, 8, 17,$ and x is equal to the mean of those five numbers?

(A) -5 **(B)** 0 **(C)** 5 **(D)** $\frac{15}{4}$ **(E)** $\frac{35}{4}$

8 Let $f(x) = x^2(1-x)^2$. What is the value of the sum

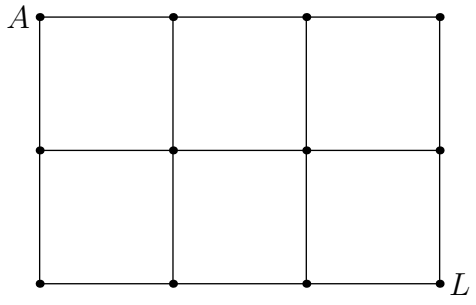
$$f\left(\frac{1}{2019}\right) - f\left(\frac{2}{2019}\right) + f\left(\frac{3}{2019}\right) - f\left(\frac{4}{2019}\right) + \cdots + f\left(\frac{2017}{2019}\right) - f\left(\frac{2018}{2019}\right)?$$

(A) 0 **(B)** $\frac{1}{2019^4}$ **(C)** $\frac{2018^2}{2019^4}$ **(D)** $\frac{2020^2}{2019^4}$ **(E)** 1

9 For how many integral values of x can a triangle of positive area be formed having side lengths $\log_2 x, \log_4 x, 3$?

(A) 57 **(B)** 59 **(C)** 61 **(D)** 62 **(E)** 63

10 The figure below is a map showing 12 cities and 17 roads connecting certain pairs of cities. Paula wishes to travel along exactly 13 of those roads, starting at city A and ending at city L, without traveling along any portion of a road more than once. (Paula *is* allowed to visit a city more than once.) How many different routes can Paula take?

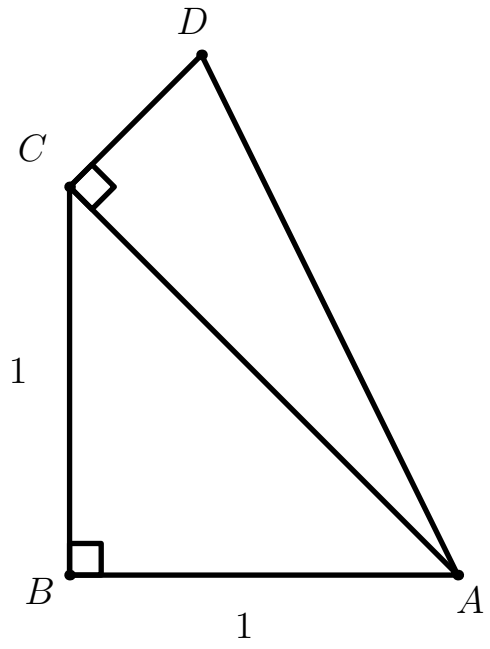


(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

11 How many unordered pairs of edges of a given cube determine a plane?

(A) 21 **(B)** 28 **(C)** 36 **(D)** 42 **(E)** 66

12 Right triangle ACD with right angle at C is constructed outwards on the hypotenuse $\overline{AC}$ of isosceles right triangle ABC with leg length 1, as shown, so that the two triangles have equal perimeters. What is $\sin(2\angle BAD)$?



(A) $\frac{1}{3}$ **(B)** $\frac{\sqrt{2}}{2}$ **(C)** $\frac{3}{4}$ **(D)** $\frac{7}{9}$ **(E)** $\frac{\sqrt{3}}{2}$

13 A red ball and a green ball are randomly and independently tossed into bins numbered with positive integers so that for each ball, the probability that it is tossed into bin k is 2^{-k} for $k = 1, 2, 3, \ldots$. What is the probability that the red ball is tossed into a higher-numbered bin than the green ball?

(A) $\frac{1}{4}$ **(B)** $\frac{2}{7}$ **(C)** $\frac{1}{3}$ **(D)** $\frac{3}{8}$ **(E)** $\frac{3}{7}$

14 Let S be the set of all positive integer divisors of $100,000$. How many numbers are the product of two distinct elements of S?

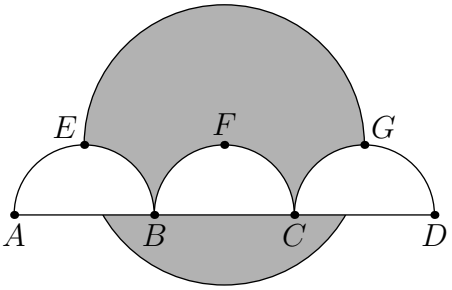
(A) 98 **(B)** 100 **(C)** 117 **(D)** 119 **(E)** 121

---

15 As shown in the figure, line segment $\overline{AD}$ is trisected by points B and C so that $AB = BC = CD = 2$. Three semicircles of radius 1, AEB, BFC, and CGD, have their diameters on $\overline{AD}$, and are tangent to line EG at E, F, and G, respectively. A circle of radius 2 has its center on F. The area of the region inside the circle but outside the three semicircles, shaded in the figure, can be expressed in the form

$$\frac{a}{b} \cdot \pi - \sqrt{c} + d,$$

where a, b, c, and d are positive integers and a and b are relatively prime. What is $a + b + c + d$?



(A) 13 **(B)** 14 **(C)** 15 **(D)** 16 **(E)** 17

---

16 There are lily pads in a row numbered 0 to 11, in that order. There are predators on lily pads 3 and 6, and a morsel of food on lily pad 10. Fiona the frog starts on pad 0, and from any given lily pad, has a $\frac{1}{2}$ chance to hop to the next pad, and an equal chance to jump 2 pads. What is the probability that Fiona reaches pad 10 without landing on either pad 3 or pad 6?

(A) $\frac{15}{256}$ **(B)** $\frac{1}{16}$ **(C)** $\frac{15}{128}$ **(D)** $\frac{1}{8}$ **(E)** $\frac{1}{4}$

---

17 How many nonzero complex numbers z have the property that $0, z$, and z^3, when represented by points in the complex plane, are the three distinct vertices of an equilateral triangle?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 4 **(E)** infinitely many

---

18 Square pyramid $ABCDE$ has base $ABCD$, which measures 3 cm on a side, and altitude $\overline{AE}$ perpendicular to the base, which measures 6 cm. Point P lies on $\overline{BE}$, one third of the way from B to E; point Q lies on $\overline{DE}$, one third of the way from D to E; and point R lies on $\overline{CE}$, two thirds of the way from C to E. What is the area, in square centimeters, of $\triangle PQR$?

(A) $\frac{3\sqrt{2}}{2}$ **(B)** $\frac{3\sqrt{3}}{2}$ **(C)** $2\sqrt{2}$ **(D)** $2\sqrt{3}$ **(E)** $3\sqrt{2}$

19 Raashan, Sylvia, and Ted play the following game. Each starts with $\$1$. A bell rings every 15 seconds, at which time each of the players who currently have money simultaneously chooses one of the other two players independently and at random and gives $\$1$ to that player. What is the probability that after the bell has rung 2019 times, each player will have $\$1$? (For example, Raashan and Ted may each decide to give $\$1$ to Sylvia, and Sylvia may decide to give her dollar to Ted, at which point Raashan will have $\$0$, Sylvia would have $\$2$, and Ted would have $\$1$, and and that is the end of the first round of play. In the second round Raashan has no money to give, but Sylvia and Ted might choose each other to give their $\$1$ to, and and the holdings will be the same as the end of the second [sic] round.

(A) $\frac{1}{7}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{1}{3}$ **(D)** $\frac{1}{2}$ **(E)** $\frac{2}{3}$

---

20 Points $A(6, 13)$ and $B(12, 11)$ lie on circle ω in the plane. Suppose that the tangent lines to ω at A and B intersect at a point on the x-axis. What is the area of ω?

(A) $\frac{83\pi}{8}$ **(B)** $\frac{21\pi}{2}$ **(C)** $\frac{85\pi}{8}$ **(D)** $\frac{43\pi}{4}$ **(E)** $\frac{87\pi}{8}$

---

21 How many quadratic polynomials with real coefficients are there such that the set of roots equals the set of coefficients? (For clarification: If the polynomial is $ax^2 + bx + c, a \neq 0$, and the roots are r and s, then the requirement is that $\{a, b, c\} = \{r, s\}$.)

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** infinitely many

---

22 Define a sequence recursively by $x_0 = 5$ and

$$x_{n+1} = \frac{x_n^2 + 5x_n + 4}{x_n + 6}$$

for all nonnegative integers n. Let m be the least positive integer such that

$$x_m \leq 4 + \frac{1}{2^{20}}.$$

In which of the following intervals does m lie?

(A) $[9, 26]$ **(B)** $[27, 80]$ **(C)** $[81, 242]$ **(D)** $[243, 728]$ **(E)** $[729, \infty]$

---

23 How many sequences of 0s and 1s of length 19 are there that begin with a 0, end with a 0, contain no two consecutive 0s, and contain no three consecutive 1s?

(A) 55 **(B)** 60 **(C)** 65 **(D)** 70 **(E)** 75

---

24 Let $\omega = -\frac{1}{2} + \frac{1}{2}i\sqrt{3}$. Let S denote all points in the complex plane of the form $a + b\omega + c\omega^2$, where $0 \leq a \leq 1, 0 \leq b \leq 1$, and $0 \leq c \leq 1$. What is the area of S?

(A) $\frac{1}{2}\sqrt{3}$ **(B)** $\frac{3}{4}\sqrt{3}$ **(C)** $\frac{3}{2}\sqrt{3}$ **(D)** $\frac{1}{2}\pi\sqrt{3}$ **(E)** π

---

25 Let $ABCD$ be a convex quadrilateral with $BC = 2$ and $CD = 6$. Suppose that the centroids of $\triangle ABC, \triangle BCD$, and $\triangle ACD$ form the vertices of an equilateral triangle. What is the maximum possible value of the area of $ABCD$?

(A) 27 **(B)** $16\sqrt{3}$ **(C)** $12 + 10\sqrt{3}$ **(D)** $9 + 12\sqrt{3}$ **(E)** 30

AIME Practice Set 2015

David Altizio

March 8, 2015

Abstract

These problems are ones I have collected from my problem-solving over the past few years that resemble AIME level problems, except that none of them have actually appeared on the AIME. Each problem (should) has a nonnegative integer answer, and each of the four sections have ten problems roughly ordered by difficulty. I tried to make the sections of similar difficulties, but this is probably not the case (Combinatorics and Number Theory seem easier than Algebra and Geometry). Have fun with the problems!

The majority of these problems have been selected from both collections of questions (AoPS, Brilliant, various math camps) and actual contests (HMMT, iTest, Math League, Mandelbrot[1], NIMO, etc.).

[1]Several of the Mandelbrot problems that appear in this collection came from the book *Mandelbrot Morsels*. To be honest, it has some pretty awesome problems, and it's one of the few books that has a bunch of problems that haven't all been released onto the Internet yet. It's definitely worth checking out!

1 Algebra

1. Let c be the larger solution to the equation $x^2 - 20x + 13 = 0$. Compute the area of the circle with center (c, c) passing through the point $(13, 7)$.

2. Suppose a, b, and c are real numbers such that

$$\left(a + \frac{1}{b}\right)\left(b + \frac{1}{c}\right)\left(c + \frac{1}{a}\right) = \left(1 + \frac{1}{a}\right)\left(1 + \frac{1}{b}\right)\left(1 + \frac{1}{c}\right).$$

   If $abc = 13$, what is $|a + b + c|$?

3. What is the only real number $x > 1$ which satisfies the equation

$$\log_2 x \log_4 x \log_6 x = \log_2 x \log_4 x + \log_2 x \log_6 x + \log_4 x \log_6 x?$$

4. Suppose that P is the polynomial of least degree with integral coefficients for which $P(\sqrt{3} + \sqrt{2}) = \sqrt{3} - \sqrt{2}$. Compute $P(2)$.

5. Let $a_1, a_2, \ldots$ be a sequence defined by $a_1 = 1$ and for $n \geq 1$,

$$a_{n+1} = \sqrt{a_n^2 - 2a_n + 3} + 1.$$

   Find a_{513}.

6. Let $\{x_n\}_{n=1}^{150}$ be a sequence of real numbers such that $x_i \in \{\sqrt{2} + 1, \sqrt{2} - 1\}$ for all positive integers i with $1 \leq i \leq 150$. For how many positive integers $1 \leq S \leq 1000$ does there exist such a sequence $\{x_n\}$ with the property that

$$x_1x_2 + x_3x_4 + x_5x_6 + \cdots + x_{149}x_{150} = S?$$

7. Let $c_1, c_2, c_3, \ldots, c_{2008}$ be complex numbers such that

$$|c_1| = |c_2| = |c_3| = \cdots = |c_{2008}| = 2011,$$

   and let $S(2008, t)$ be the sum of all products of these 2008 complex numbers taken t at a time. Let Q be the maximum possible value of

$$\left|\frac{S(2008, 1492)}{S(2008, 516)}\right|.$$

   Find the remainder when Q is divided by 1000.

8. Let S be the sum of all x such that $1 \leq x \leq 99$ and

$$\{x^2\} = \{x\}^2.$$

   Find the number formed by the first three digits of $\lfloor S \rfloor$. (Here $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x and $\{x\} = x - \lfloor x \rfloor$ denotes the fractional part of x.)

9. Let a, b, c, and d be positive real numbers such that

$$\begin{aligned} a^2 + b^2 &= c^2 + d^2 = 2008, \\ ac &= bd = 1000. \end{aligned}$$

   If $S = a + b + c + d$, compute the value of $\lfloor S \rfloor$.

10. Suppose a, b, c, d are integers such that

$$a + b + c + d = 0 \qquad \text{and} \qquad (ab - cd)(ac - bd)(ad - bc) + 528^2 = 0.$$

    What is the maximum possible value of a?

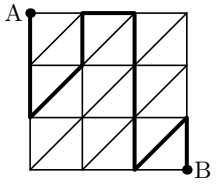
2 Combinatorics

1. A random pizza is made by flipping a fair coin to decide whether to include pepperoni, then doing the same for sausage, mushrooms, and onions. The probability that two random pizzas have at least one topping in common can be written in the form $\frac{m}{n}$ where m and n are positive integers. Find $m+n$.

2. How many permutations $(a_1, a_2, \ldots, a_{10})$ of $1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ satisfy

$$|a_1 - 1| + |a_2 - 2| + \cdots + |a_{10} - 10| = 4?$$

3. How many paths are there from A to B through the network shown if you may only move up, down, right, and up-right? A path also may not traverse any portion of the network more than once. A sample path is highlighted.



4. Consider the set $S = \{1, 2, 3, 4, 5, \ldots, 100\}$. How many subsets of this set with two or more elements satisfy:

   (i) the terms of the subset form an arithmetic sequence, and

   (ii) we cannot include another element from S with this subset to form an even longer arithmetic sequence?

5. Simon and Garfunkle play in a round-robin golf tournament. Each player is awarded one point for a victory, a half point for a tie, and no points for a loss. Simon beat Garfunkle in the first game by a record margin as Garfunkle sent a shot over the bridge and into troubled waters on the final hole. Garfunkle went on to score 8 total victories, but no ties at all. Meanwhile, Simon wound up with exactly 8 points, including the point for a victory over Garfunkle. Amazingly, every other player at the tournament scored exactly n. Find the sum of all possible values of n.

6. An ant starts at the origin of a coordinate plane. Each minute, it either walks one unit to the right or one unit up, but it will never move in the same direction more than twice in the row. In how many different ways can it get to the point (5,5)?

7. Thaddeus is given a 2013×2013 array of integers each between 1 and 2013, inclusive. He is allowed two operations:

   1. Choose a row, and subtract 1 from each entry.
   2. Chooses a column, and add 1 to each entry.

   He would like to get an array where all integers are divisible by 2013. Let M be the number of initial arrays for which this is possible. What is the number formed by the last three digits of M?

8. If you flip a fair coin 1000 times, let P be the expected value of the product of the number of heads and the number of tails. What are the first three digits of P?

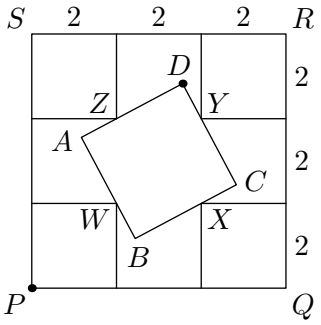
9. Suppose N is the number of ways to partition the counting numbers from 1 to 12 (inclusive) into four sets with three numbers in each set so that the product of the numbers in each set is divisible by 6. What is the number formed by the last three digits of N?

10. Seven points are spaced equally around a circle each having labeled with some number. A labeling is *clean* if for any two pairs of points a, b and c, d with a having the same label as b and c as d, but a not having the same label as c, the chords connecting ab and cd do not intersect. Additionally, two *clean* labelings are the same if the set of points that have the same label in one labeling are the same as in the other and if the points can be rotated to equal the other. How many unique *clean* labelings are there?

3 Geometry

1. Regular hexagon $ABCDEF$ is given in the plane. If the area of the triangle whose vertices are the midpoints of $\overline{AB}$, $\overline{CD}$, and $\overline{EF}$ is 225, what is the area of $ABCDEF$?

2. In the corners of a square $PQRS$ with side length 6 cm four smaller squares are placed with side lengths 2 cm. Let us denote their vertices by W, X, Y, Z like in the picture. A square $ABCD$ is constructed in such a way, that points W, X, Y, Z lie inside the sides AB, BC, CD, DA respectively. Find the square of the largest possible distance between points P and D.



3. Two perpendicular planes intersect a sphere in two circles. These circles intersect in two points, A and B, such that $AB = 42$. If the radii of the two circles are 54 and 66, find the remainder when R^2 is divided by 1000, where R is the radius of the sphere.

4. Two circles, ω_1 and ω_2, have radii of 5 and 12 respectively, and their centers are 13 units apart. The circles intersect at two different points P and Q. A line l is drawn through P and intersects the circle ω_1 at $X \neq P$ and ω_2 at $Y \neq P$. Find the maximum value of $PX \cdot PY$.

5. Two circles in the Cartesian plane have four common tangent lines. If the slopes of these lines are 2, 3, 4, and m, in increasing order, then calculate $\lfloor 100m \rfloor$.

6. Let point O be the origin of a three-dimensional coordinate system, and let points A, B, and C be located on the positive x, y, and z axes, respectively. Suppose $OA = \sqrt[4]{75}$ and $m\angle BAC = 30^\circ$. Compute $100K$, where K is the area of $\triangle ABC$.

7. Let $\triangle ABC$ be an isosceles triangle with $AB = AC$, and denote by ω the unique circle inscribed inside the triangle. Suppose the orthocenter of $\triangle ABC$ lies on ω. Then there exist relatively prime positive integers m and n such that $\cos \angle BAC = \frac{m}{n}$. Find $m + n$.

8. Let $\triangle ABC$ have $AB = 6$, $BC = 7$, and $CA = 8$, and denote by ω its circumcircle. Let N be a point on ω such that AN is a diameter of ω. Furthermore, let the tangent to ω at A intersect BC at T, and let the second intersection point of NT with ω be X. The length of $\overline{AX}$ can be written in the form $\frac{m}{\sqrt{n}}$ for positive integers m and n, where n is not divisible by the square of any prime. Find $m + n$.

9. Given a convex, n-sided polygon P, form a $2n$-sided polygon $\text{clip}(P)$ by cutting off each corner of P at the edges' trisection points. In other words, $\text{clip}(P)$ is the polygon whose vertices are the $2n$ edge trisection points of P, connected in order around the boundary of P. Let P_1 be an isosceles trapezoid with side lengths $13, 13, 13$, and 3, and for each $i \geq 2$, let $P_i = \text{clip}(P_{i-1})$. This iterative clipping process approaches a limiting shape $P_\infty = \lim_{i \to \infty} P_i$. If the difference of the areas of P_{10} and P_∞ is written as a fraction $\frac{x}{y}$ in lowest terms, calculate the number of positive integer factors of $x \cdot y$.

10. Let ABC be a triangle, and I its incenter. Let the incircle of ABC touch side BC at D, and let lines BI and CI meet the circle with diameter AI at points P and Q, respectively. Given $BI = 6, CI = 5, DI = 3$, find the sum of the numerator and denominator of $(DP/DQ)^2$ when written in lowest terms.

4 Number Theory

1. Jack chose nine different integers from 1 through 19 and found their sum. From the remaining ten integers, Jill chose nine and found their sum. If the ratio of Jack's sum to Jill's sum was $7 : 15$, which of the nineteen integers was chosen by *neither* Jack nor Jill?

2. There exist unique positive integers x and y such that $4^y - 615 = x^2$. What is the value of $x + y$?

3. In the binary expansion of
$$\frac{2^{2007} - 1}{2^{225} - 1},$$
how many of the first $10,000$ digits to the right of the radix point are 0's?

4. For positive integers $n \geq 2$, define $g(n)$ to be one more than the largest proper divisor of n. Hence $g(35) = 8$, since the proper divisors of 35 are 1, 5, and 7. For how many n in the range $2 \leq n \leq 100$ do we have $g(g(n)) = 2$?

5. A positive integer n is called a *good* number if
$$n^3 + 7n - 133 = m^3$$
for some positive integer m. What is the sum of all good numbers?

6. How many zeroes occur at the end of the number $1999^6 + 6 \cdot 1999 + 5$?

7. All the digits of the positive integer N are either 0 or 1. The remainder after dividing N by 37 is 18. What is the smallest number of times that the digit 1 can appear in N?

8. It is well-known that the n^{th} triangular number can be given by the formula $n(n+1)/2$. A Pythagorean triple of *square numbers* is an ordered triple (a, b, c) such that $a^2 + b^2 = c^2$. Let a Pythagorean triple of *triangular numbers* (a PTTN) be an ordered triple of positive integers (a, b, c) such that $a \leq b < c$ and
$$\frac{a(a+1)}{2} + \frac{b(b+1)}{2} = \frac{c(c+1)}{2}.$$
For instance, $(3, 5, 6)$ is a PTTN $(6 + 15 = 21)$. Here we call both a and b *legs* of the PTTN. Find the smallest natural number n such that n is a leg of *at least* six distinct PTTNs.

9. How many of the first 2010 rows of Pascal's Triangle (rows 0 through 2009) have exactly 256 odd entries?

10. For how many integers $1 \leq n \leq 9999$ is there a solution to the congruence
$$\phi(n) \equiv 2 \pmod{12},$$
where $\phi(n)$ is the Euler phi-function?

TRIG PREPARATION PART 1

1. Helpful Formulas

This page describes some of the basic formulas and identities for relating sine and cosine functions. Together with knowledge of a few of the standard angles from the special right triangles, these equations can be used to derive and solve a wide variety of problems.

1.1. Pythagorean Identity.

$$\sin(x)^2 + \cos(x)^2 = 1$$

1.2. Parity and Shifts.

$$\begin{aligned}
\sin(-x) &= -\sin(x) \\
\cos(-x) &= \cos(x) \\
\sin(90 - x) &= \cos(x) \\
\cos(90 - x) &= \sin(x) \\
\sin(180 - x) &= \sin(x) \\
\cos(180 - x) &= \cos(x)
\end{aligned}$$

1.3. Angle Sums and Differences.

$$\begin{aligned}
\sin(x + y) &= \sin(x)\cos(y) + \cos(x)\sin(y) \\
\sin(x - y) &= \sin(x)\cos(y) - \sin(y)\cos(x) \\
\cos(x + y) &= \cos(x)\cos(y) - \sin(x)\sin(y) \\
\cos(x - y) &= \cos(x)\cos(y) + \sin(x)\sin(y)
\end{aligned}$$

1.4. Doubled and Halved Angles.

$$\begin{aligned}
\sin(2x) &= 2\sin(x)\cos(x) \\
\cos(2x) &= \cos^2(x) - \sin^2(x) \\
\sin(\frac{x}{2}) &= \pm\sqrt{\frac{1 - \cos(x)}{2}} \\
\cos(\frac{x}{2}) &= \pm\sqrt{\frac{1 + \cos(x)}{2}}
\end{aligned}$$

1.5. Product to Sum Identities.

$$\begin{aligned}
\cos(x)\cos(y) &= \frac{1}{2}(\cos(x + y) + \cos(x - y)) \\
\sin(x)\sin(y) &= \frac{1}{2}(\cos(x - y) - \cos(x + y)) \\
\sin(x)\cos(y) &= \frac{1}{2}(\sin(x + y) + \sin(x - y))
\end{aligned}$$

1.6. Sum to Product Identities.

$$\begin{aligned}
\cos(x) + \cos(y) &= 2\cos(\frac{x + y}{2})\cos(\frac{x - y}{2})) \\
\cos(x) - \cos(y) &= -2\sin(\frac{x + y}{2})\sin(\frac{x - y}{2})) \\
\sin(x) + \sin(y) &= 2\sin(\frac{x + y}{2})\cos(\frac{x - y}{2})) \\
\sin(x) - \sin(y) &= 2\sin(\frac{x - y}{2})\cos(\frac{x + y}{2}))
\end{aligned}$$

1.7. Law of Cosines.

$$\begin{aligned}
a^2 &= b^2 + c^2 - 2bc\cos(A) \\
b^2 &= a^2 + c^2 - 2ac\cos(B) \\
c^2 &= a^2 + b^2 - 2ab\cos(C)
\end{aligned}$$

1.8. Law of Sines.

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

Date: June 9, 2017.

2. Triangle Warmups

(1) Three semicircles have their diameters on the sides of right triangle ABC. The area of the small semi circle is 5π and the area of the medium semicircle is 7π. What is the sum of the areas of all three semicircles?

(2) A 20 foot ladder rests against a wall so the top of the ladder is 19 feet off the ground. If the top of the ladder drops 6 feet, how far did the bottom of the ladder move?

(3) A square and an equilateral triangle have the same perimeter. What is the ratio of the area of the circle circumscribed about the square to the area of the circle circumscribed about the triangle?

(4) A square and an equilateral triangle have the same area. What is the ratio of the area of the circle circumscribed about the square to the area of the circle circumscribed about the triangle?

(5) Use your favorite 30–60–90 and 45–45–90 triangles to determine the 6 trig function values for $\frac{\pi}{3}$, $\frac{\pi}{4}$, and $\frac{\pi}{6}$.

(6) Sketch the function $5\cos(\frac{x}{2}) - 1$ below.

(7) From the function $2\sin(\pi x + 3) + 4$ determine the amplitude, period, phase shift, and vertical shift.

3. Trig Identities

(1) How many solutions for x are there of $\sin(x)\tan(x) + 2x\sec(x) = -\cos(x)$?

(2) Use the angle sum identities to compute $\sin(105)$, $\sin(15)$, $\cos(75)$, and $\cos(15)$.

(3) What is $\cos(20) \cdot \cos(40) \cdot \cos(80)$?

(4) Let w, x, y, z satisfy $w^2 + x^2 = y^2 + z^2 = 1$ and $wy - xz = \frac{1}{2}$. What is $wz + xy$?

(5) Compute $\sin(1)$ times the average of $2\sin(2), 4\sin(4), \ldots, 178\sin(178), 180\sin(180)$.

(6) In triangle ABC we know that $AB = 3$, $BC = 5$ and $AC = 7$. What in the measure of angle ABC?

(7) Consider two triangles, ABC and DEF with $AB = DE = 7$, $BC = EF = 5$, and $\angle CAB = \angle FDE = 30$. Use the law of sines to show that $\sin(\angle ACB) = \sin(\angle DFE)$. Must the two triangles by congruent?

1. A slanted ladder, x unit long, has its top resting on a vertical wall. Suppose the bottom of the ladder (resting on the horizontal ground) is shifted 14 units to the wall, then top of the ladder will move 8 units up the wall. Suppose instead that the bottom of the ladder is shifted 21 units away from the wall, then top of the ladder will move 27 units down the wall. How long is the ladder?

   a) 65 b) 40 c) 81 d) 100 e) 125

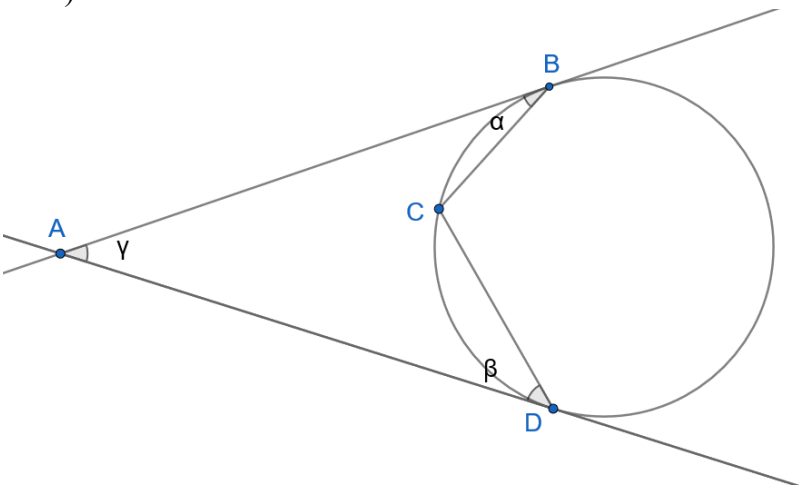
2. $\overline{SMC}_{10}$, $\overline{SCM}_{10}$ and $\overline{CMS}_{10}$ are all perfect squares, where S, M and C are digits (from 0 to 9) of the decimal representation of the numbers. Given that S, M and C are not necessarily distinct, what is the maximum value of $S + M + C$?

   a) 9 b) 12 c) 16 d) 20 e) 27

3. 10 boxes are placed in a row. In how many ways can we put 5 balls, each of a different colour into these boxes if each box can hold at **most** 1 ball and no 2 boxes without balls are adjacent?

   a) 1 b) 4 c) 5 d) 6 e) 252

4. Let $\alpha = A\hat{B}C, \beta = A\hat{D}C, \gamma = B\hat{A}D$ in the diagram below. Given $\alpha + \beta \leq \gamma$, what is the minimum value of γ? (Quadrilateral $ABCD$ is not convex and AB and AD are tangent lines).



   a) 30° b) 45° c) 60° d) 75° e) 90°

5. Simplify $512^{\log_{32} 2023}$.

   a) 2^{2023} b)16^{2023} c) 2023^4 d) 2023^{16} e) $2023^{9/5}$

6. What is the probability that in a class of 18 people, there exists a group of 3 people born on the same day of the week?

   a) 0 b) 1/6 c) 5/24 d) 20/21 e) 1

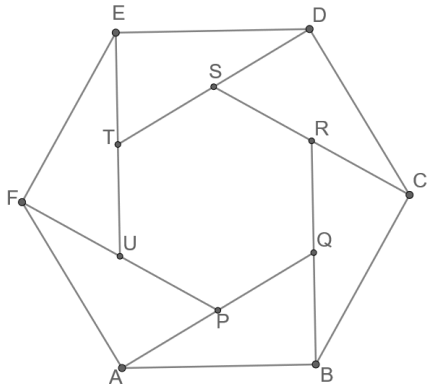
7. In triangle ABC with $|BC| = 72$, let D be the midpoint of $|BC|$ and let E be a point on $|AD|$ such that $|BE| = |BD|$. Suppose E lies in the interior of ABC such that $\angle EBA = \angle DAC$, $\angle EBC = \angle BCA$. Find the length of the altitude from A to BC.

   a) 32 b) 36 c) $24\sqrt{3}$ d) 54 e) $18\sqrt{7}$

8. Find the number of positive integers that are factors of $3^{19}.7^{12}.10^{25}$ but are multiples of $3^{15}.7^{10}.10^{19}$.

a) 105 b) 735 c) 3240 d) 105360 e) 175760

9. Let $ABCDEF$ be a regular hexagon. Suppose $PQRSTU$ is a hexagon such that $AP = PQ$, $BQ = QR$, $CR = RS$, $DS = ST$, $ET = TU$ and $FU = UP$, as illustrated (P lies on AQ, Q lies on BR, R lies on CS, S lies on DT, T lies on EU, and U lies on FP). Find the ratio of the area of the bigger hexagon to the smaller one.



a) 2 b) 3 c) 4 d) $2+\sqrt{3}$ e) $2\sqrt{3}$

10. Let a be a zero of the function $f(x) = x^2 - x + 1$. Find the value of $a^{2023} + \left(\frac{1}{a}\right)^{2023}$.

a) 1 b) 2 c) 2023 d) -1 e) -2

11. A circle of radius 10 with its center at the origin is drawn in the $x - y$ plane. Find the number of lattice points that are on or inside the circle.

a) 31 b) 37 c) 173 d) 317 e) 371

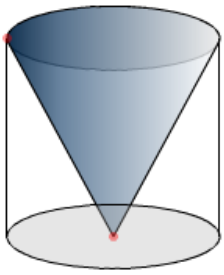
12. Given positive integers $a_1 \le a_2 \le a_3 \le a_4$, let $F(a_1, a_2, a_3, a_4)$ denote the number of ways of filling a 4 by 4 grid with $a_1 + a_2 + a_3 + a_4$ dots in such a way that each grid contains at most 1 dot and both row k and column k contain exactly a_k dots. Suppose that $F(1,2,2,3) = 24, F(1,2,3,3) = 13$ and $F(1,2,3,4) = 1$. Find $F(1,1,2,3)$.

a) 1 b) 12 c) 13 d) 24 e) 34

13. What is the remainder when $1^1 + 2^2 + 3^3 + \cdots + 2023^{2023}$ is divided by 7?

a) 0 b) 1 c) 2 d) 3 e) 4

14. A cylinder is filled to water till it is $\frac{11}{24}$ full. A cone of same height and circular base is submerged in the cylinder as illustrated. What is the height of the water now, as a fraction of the height of the cylinder?



a) ½ b) 1 c) $^{11}/_{24}$ d) $^{5}/_{6}$ e) $^{5}/_{8}$

15. Find the number of integers greater than **6200** that is formed from the digits 1, 3, 6, 8 and 9, where each digit is used at **most** once.

a) 54 b) 66 c) 120 d) 186 e) 240

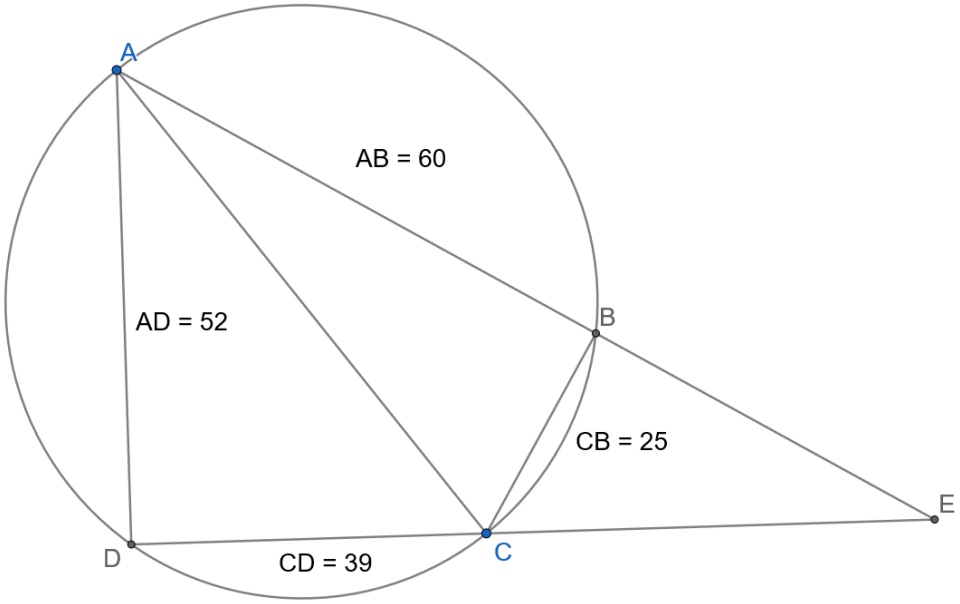
16. Three sequences, a_n, b_n and c_n satisfy $a_n = a_{n+1}a_{n-1}$, $b_n + 1 = b_{n+1}b_{n-1}$ and $c_n = a_{n+1}b_{n-1}$ for all integers n. Find the least value for $T > 0$ such that we **must** have $c_{n+T} = c_n$ for all n.

a) 3 b) 5 c) 6 d) 15 e) 30

17. Which of the following is **not** a factor of $t^{12} - t^7 + t - 1$?

a) $t + 1$ b) $t^2 - t + 1$ c) $t^2 + t + 1$ d) $t^4 - t + 1$
e) $t^6 - t + 1$

18. What is the perimeter of triangle BCE in the diagram below? ($ABCD$ is cyclic)



a) 104 b) 3200/33 c) 176 d) 350/3 e) 1000/11

19. How many ordered pairs of positive integers (m, n) satisfy,

$$\frac{1}{m!} + \frac{1}{n!} = \frac{1}{lcm(m,n)}?$$

a) 1 b) 2 c) 3 d) 4 e) 5

20. What is the sum of the coefficients of terms with even powers of x (terms of the form $x^{2m}y^n$, m and n are integers) in the expansion of $(1 - 2x + y)^{2023}$?

a) 0 b) 2^{4045} c) 2^{4046} d) 2^{2023} e) 2^{2022}

21. Super Eagles coach is tasked with carrying a squad of 23 men for the FIFA's world cup tournament. This squad must consist of at least one goalkeeper, one defender, one midfielder and one attacker. Given that the number of defenders must not be less than the number of midfielders which also must not be less than the number of attackers which also must not be less than the number of goalkeepers, how many possible squad formations are possible? (A possible formation would be 2 goalkeepers, 8 defenders, 7 midfielders and 6 attackers)

a) 94 b) 64 c) 1540 d) 1771 e) 8855

22. An octagon of perimeter 32 units is such that 4 sides have equal lengths of 3 units and the remaining 4 sides have equal lengths of 5 units as well. Order does not matter. Given that this octagon is cyclic, find its circumradius.

a) 4 b) 5 c) $\frac{4+\sqrt{15}}{2}$ d) $\sqrt{\frac{34+15\sqrt{2}}{2}}$ e) $\sqrt{\frac{15+20\sqrt{2}}{2}}$

23. Let a, b, c be positive real numbers such that,

$$a + b + c = 5 \text{ and } \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 2$$

The range of possible values of c is the real number interval $[e, f]$. Find the value of $f - e$.

a) 1 b) ½ c) $3/2$ d) ¾ e) $5/4$

24. How many quadruples of positive integers (d_1, d_2, d_3, d_4) are there such that $d_1|5!$, $d_2|5!$, $d_3|5!$, $d_4|5!$ and $gcd(d_1, d_2, d_3, d_4) = 1$? ($a|b$ means a divides b)

a) $2^4 \cdot 3^2 \cdot 5 \cdot 7$ b) $3^2 \cdot 5^4 \cdot 7$ c) $2^3 \cdot 3 \cdot 5$ d) $2^3 \cdot 3^2 \cdot 5 \cdot 7$ e) 2^{16}

25. For all $\binom{2023}{i} = 3^{a_i} \cdot b_i$, $\gcd(b_i, 3) = 1$ with a_i, b_i integers for $i = 0, 1, 2, \ldots, 2023$. Find the largest value of i that maximises a_i. ($\binom{n}{k}$ represents the number of ways of selecting k items from n items, $\gcd(a, b)$ represents the highest common factor of positive integers a and b)

a) 1943 b) 728 c) 729 d) 1376 e) 1457

Instructions:

1. Duration is 4 hours.
2. Use white plane A4 papers for your work and write your name on each paper you submit.
3. Proofs should be provided for all lemmas used and all theorems used should be properly named and stated.
4. Ask questions about any problem you do not understand within the first 30 minutes of the exam.

Problem 1: $ABCD$ is a quadrilateral with,

$$|AB| = |AD|,\ \cos\left(\frac{1}{2}\angle BAD + \angle DCB\right) = 0 \text{ and } \frac{1}{\sqrt{3}} \leq \frac{|AC|}{|AB|} \leq \sqrt{3}.$$

What is the minimum possible value of $\angle BAD$?

Problem 2: Find all triples of primes (p, q, r), such that

$$\begin{cases} p^q r^2 + q^r p^2 = r^p q^2 + 13 \\ pr^2 + qp^2 = rq^2 + 73 \end{cases}$$

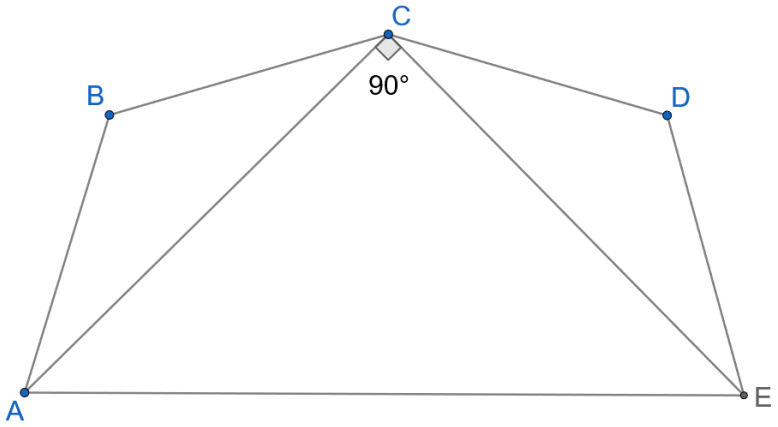
Problem 3: In a gathering of people, if 2 people know each other, we say there is a ***connection*** between them. ***Connection*** is mutual. If 3 people all know each other, we say they are a ***3clique***. Given that there are 7 people in a gathering, determine with proof, the minimum number of ***connections*** that guarantees the existence of a ***3clique***.

Problem 4: Find all functions $f: \mathbb{Q} \longrightarrow \mathbb{R}$ such that there exist functions $g: \mathbb{Q} \longrightarrow \mathbb{R}$, $h: \mathbb{Q} \longrightarrow \mathbb{R}$ with $f(1) = 1$, $h(0) \neq 0$ and

$$g(x + y) + g(x - y) - 2g(x) = h(x)f(y) \forall x, y \in \mathbb{Q}$$

1st Round

1. From midnight of a certain day, the minute hand of a bewitched clock begins to misbehave in that it moves at same rate but in the opposite direction, while the hour and seconds hands behave normally. When next, after midnight, does the bewitched clock tell the correct time?
   A)06:00 B) 12:00 C) 15:15 D) 06:30 E) 00:30
2. How many ways can the letters in the word $OLYMPIAD$ be rearranged so that all the vowels form a string of adjacent characters? (Example $LYMPIAOD$)
   A)4320 B) 40320 C) 43200 D) 4032 E) 432
3. How many 4-digit palindromes are divisible by 7? (A palindrome is a number of the form $abba$)
   A) 7 B) 9 C) 18D) 19 E) 20
4. Given $ABCDE$ is a convex pentagon with $\angle ABC = \angle CDE = 120°$, $\angle ACE = 90°$ and $|AB| = |BC| = |CD| = |DE| = 1$. Find |AE|.

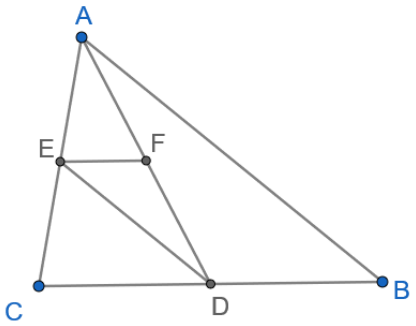


   A)2 B) $\sqrt{6}$ C) $2\sqrt{2}$ D) $1+\sqrt{3}$ E) 3
5. Find the last digit of $1^{2^{3^4}} + 2^{3^{4^1}} + 3^{4^{1^2}} + 4^{1^{2^3}}$.
   A)0 B) 2 C) 4 D) 6 E) 8

6. Given that $8!\,7!\,6!\,5!\,4!\,3!\,2!\,1! = 2^a \cdot b$, where a and b are positive odd integers. What is the value of a?

   A)7 B) 15 C) 23 D) 25 E) 45

7. Given that D is the midpoint of line segment BC, E is the midpoint of line segment AC and F is the midpoint of line segment AD, find the ratio of the area of triangle ABC to triangle DEF.



   A)4 B) 6 C) 8 D) $\frac{1}{4}$ E) $\frac{1}{8}$

8. For which of the following pairs $(x,\ y)$ does the inequality $12x^2 + 6y^2 \geq 17xy$ fail?

   A)$(3,\ 5)$ B) $(4,\ 5)$ C) $(4,\ 7)$ D) $(5,\ 7)$ E) $(6,\ 7)$

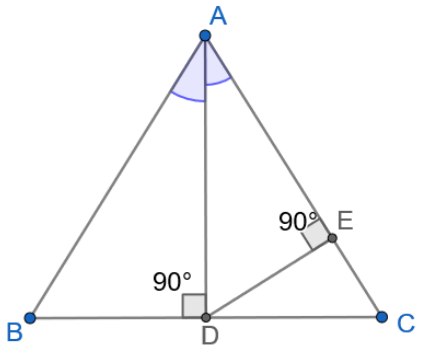
9. Let p be a prime **greater than** 7, then $(2p - 2)!!\ (mod\ p)$ is? ($x!!$ is double factorial given by $x!! = x(x-2)(x-4)\ldots 3\cdot 1$, if x is odd and $x!! = x(x-2)(x-4)\ldots 4\cdot 2$, if x is even)

   A)0 B) 1 C) 2 D) $p - 2$ E) $p - 1$

10. 10 friends arrive at a hotel to spend the night, but the hotel has only 4 rooms free: two 2-man rooms and two 3-man rooms (Rooms are not identical). Given the 10 friends decide to spend the night in the given rooms, in how many ways can they book the rooms?

    A) 25200 B) 6300 C) 144 D) 10! E) None of the above

11. In the figure, $|AB| = 25$, $|CE| = 9$, $\angle BAD = \angle DAE$, $AD \perp BC$ at D and $DE \perp AC$ at E. Find $|BC|$.



A) 25 B) 29 C) 30 D) 34 E) Cannot be found

12. Sum of two irrational numbers is 1 less than their product, and 8 less than their sum of squares. Find the larger of the two numbers.

A) $-1-\sqrt{2}$ B) 2 C) 3 D) $-1+\sqrt{2}$ E) $1+\sqrt{2}$

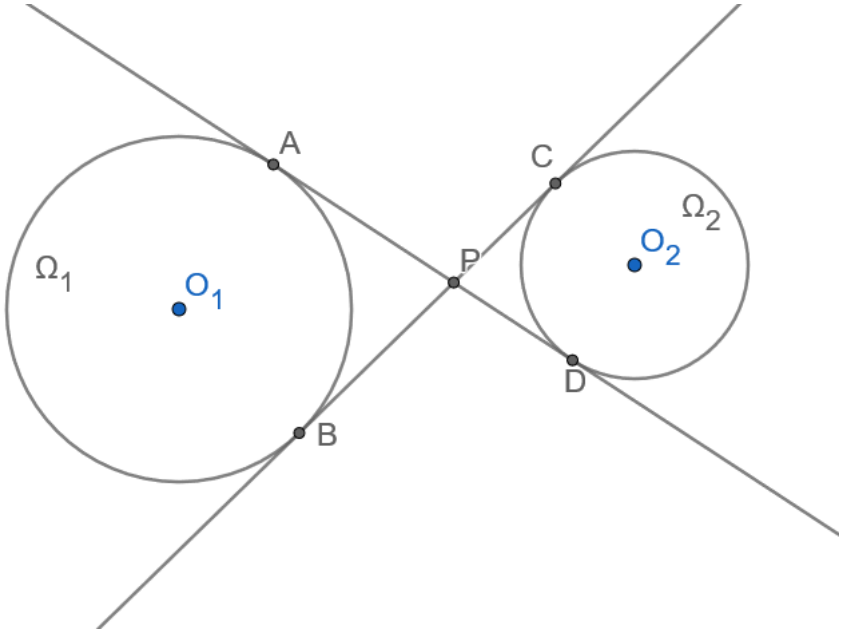
13. In an Olympiad class, half of the students love Geometry (G), half love Number Theory (NT) and half love Combinatorics (C). They all love Algebra. It is also known that 6 love C and NT, 7 love C and G, 8 love G and NT, and 14 love NT but not C nor G. How many students love G but not NT nor C?

A) 12 B) 13 C) 14 D) 15 E) 16

14. An integer n is said to be $k-lengthy$ if it can be written as the sum of k consecutive positive integers. For example, 6 is $3-lengthy$ as $6=1+2+3$. Suppose that there exists a 3-digit positive integer, m, such that m is $k-lengthy$ and k is as large as possible, find the value of $m+k$.

A) 1079 B) 945 C) 990 D) 1011 E) 1034

15. Let AD and BC be common tangents to circles Ω_1 and Ω_2. Let O_1 and O_2 be the centers of the Ω_1 and Ω_2 respectively. Given that $|O_1O_2|=15$, $|O_1A|=6$, $|O_2C|=3$ and AD intersects BC at P, find $|BP|$.



A)5 B) 10 C) 7 D) 6 E) 8

16. Consider the sequence of positive integers defined by $a_{n+1} = na_n$ for integers $n \geq 1$. If $a_1 \leq 1000$, find the sum of all the possible number of zeroes a_{2022} can end with.

A)1512 B) 1007 C) 2018 D) 2525 E) 503

17. A bag contains x blue, y red and z yellow identical balls. Gauss picks 3 balls at random with **replacement**. Given the first two balls are of different colours, what is the probability that the third ball is also of a different colour from the first two?

A) $\frac{1}{3}$ B) $\frac{x!y!z!}{(x+y+z)!}$ C) $\frac{xy+yz+zx}{9xyz}$ D) $\frac{3}{(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)}$

E) $\frac{6xyz}{(x+y+z)^3}$

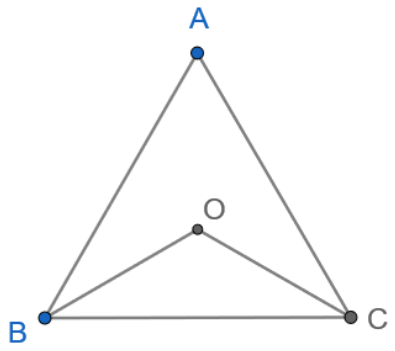
18. Let x, y and n be integers satisfying $nx + (n-1)y = 1$ and $n > 3$. Then, all of the following are possible except

A) $x = 2n - 1$ B) $y = 2n - 1$ C) $y - x = 2n - 3$ D) $2x + y = 2n - 3$ E) $2y + x = 2n - 3$

19. Given that $f: R \longrightarrow R$ is a non-constant linear function and there exist $a, b \in R$ such that $f(a-b) = a$, $f(a) = b$, $f(b) = a + b - 1$ and $f(a+b) = 1$. Find $f(1)$.

A)1 B) -3 C) 5 D) 7 E) -9

20. $\angle BAC = \angle CBO = \frac{1}{2}\angle BOC = 50°$, $|AB| = |BC|$. Which of the following is correct?



A)O is the circumcenter of ABC. B) $\angle ABC = 60°$ C) $BO \perp AC$

D) Area of ABC is thrice that of BOC. E) $|AC| = |BO| + |OC|$.

21. John is playing a game with three standard dice. In a single move, he tosses the dices and records the number displayed by each of the dice. John wins the game if the three numbers can be placed side by side to form a 3-digit number divisible by 11. Suppose the probability that John wins is $\frac{p}{q}$ where the fraction $\frac{p}{q}$ is in lowest term. Find $p + q$.
A)11 B) 5 C) 31D) 29 E) 77

22. Let the polynomial $p(x) = 5x^3 + 3x^2 - 10$ have roots a, b and c. What is the value of $\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}$?
A) $\frac{723}{250}$ B) $-\frac{723}{250}$ C) $\frac{27}{25}$ D) $-\frac{27}{25}$ E) Not possible to be determined

23. 3 positive integers **greater than** 200 are such that the H.C.F of the 1st and 2nd is 143, H.C.F of the 2nd and 3rd is 34, and the H.C.F of the 3rd and 1st is 1. Given the L.C.M of the 3 positive integers is 578578, find the smallest of the 3 positive integers. (H.C.F also known as G.C.D is the Highest Common Factor or Greatest Common Divisor, and L.C.M is the Lowest Common Multiple)
A)4862 B) 578 C) 286 D) 338 E) 442

24. Let S denote the set of positive multiples of 12, 15 or 20 that are less than 2022. Find the sum of all the elements of S.
A)409047 B) 403 C) 341727 D) 337 E) 289020

25. Given $P_1P_2P_3 \ldots P_{2022}$ is a convex polygon, what is the value of the following summation?

$$\sum_{i=1}^{i=2022} \angle P_i P_{i+5} P_{i+10}$$

where $P_{2023} = P_1$, $P_{2024} = P_2$, $P_{2025} = P_3$ and so on.

A) $1006 \bullet 360$ B) $2012 \cdot 360$ C) $2013 \cdot 180$ D) $2017 \cdot 360$ E) Not enough information.

Instructions in addition to the ones given in Finalists' Orientation meeting

1. Duration is 4 hours.
2. Use white plane A4 papers for your work and write your name on each paper you submit.
3. Proofs should be provided for all lemmas used and all theorems used should be properly named and stated.

Problem 1: Let $n \in \mathbb{N}$. Prove that

$$\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sqrt{x_i}+\sqrt{y_i}} \geq \frac{\sqrt{n}}{\sqrt{\sum_{i=1}^{n} x_i}+\sqrt{\sum_{i=1}^{n} y_i}}$$

$\forall x_i, y_i \in \mathbb{R}^+, i = 1, 2, \ldots, n.$

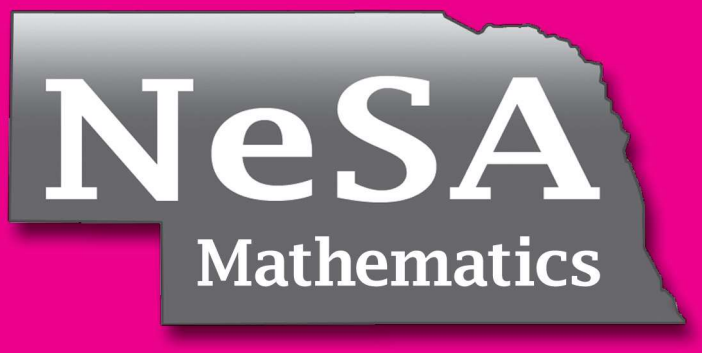
Problem 2: Find all triples of prime numbers (p, q, r), such that $q|(r-1)$ and

$$\frac{r(p^{q-1}-1)}{q^{p-1}-1}$$

Is prime.

Problem 3: The numbers 2, 3, 4, …, 100 are written on a board. Chibuike and Ismail play a game of erasing numbers from the board using the following rule: If the number, a, is erased, then only numbers, b, such that $gcd(a, b) = 1$, can be erased, and the person that erased last wins. If Chibuike starts the game, does there exist a winning strategy for him? (Determine with proof.)

Problem 4: Let Ω be a circle with center O and P a point outside it. Tangents from P are drawn to touch the circle at A and B. A point, T, is arbitrarily chosen on major arc AB, and D is the foot of T on AB. K, L, M, N are the midpoints of TA, TB, TD, AB respectively. PT intersects MN at point S. Lines l_a and l_b are the reflections of OA and OB over the angle bisectors of $\angle SAL$ and $\angle SBK$, respectively. Show that l_a, l_b and TD are concurrent.



Nebraska State Accountability

Grade 11

Mathematics

Practice Test

Name:

Nebraska Department of Education 2010

Directions:

On the following pages are multiple-choice questions for the Grade 11 Practice Test, a practice opportunity for the *Nebraska State Accountability–Mathematics (NeSA–M).*

Each question will ask you to select an answer from among four choices.

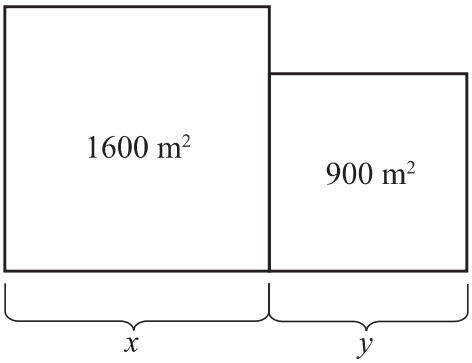
For all questions:

- Read each question carefully and choose the best answer.
- You may use scratch paper to solve the problems.
- The Mathematics Reference Sheet is provided in the back of the test booklet. You may refer to this page any time during the test.
- You may not use a calculator on this test.
- Be sure to answer ALL the questions.

Remember only one of the answers provided is the correct response.

STOP.

1. **Use the diagram below to answer the question.**



The area of each square is given in the diagram. What is the value of $x + y$?

A. 30 meters

B. 40 meters

C. 70 meters

D. 120 meters

2. What is the value of the expression $-4\sqrt{25} + 4^3$?

A. 32

B. 44

C. 84

D. 112

3. What is the simplified form of the expression $(6x^4 + 4x^3 - 2x^2 + 5) - (3x^4 - 2x^3 + x + 4)$?

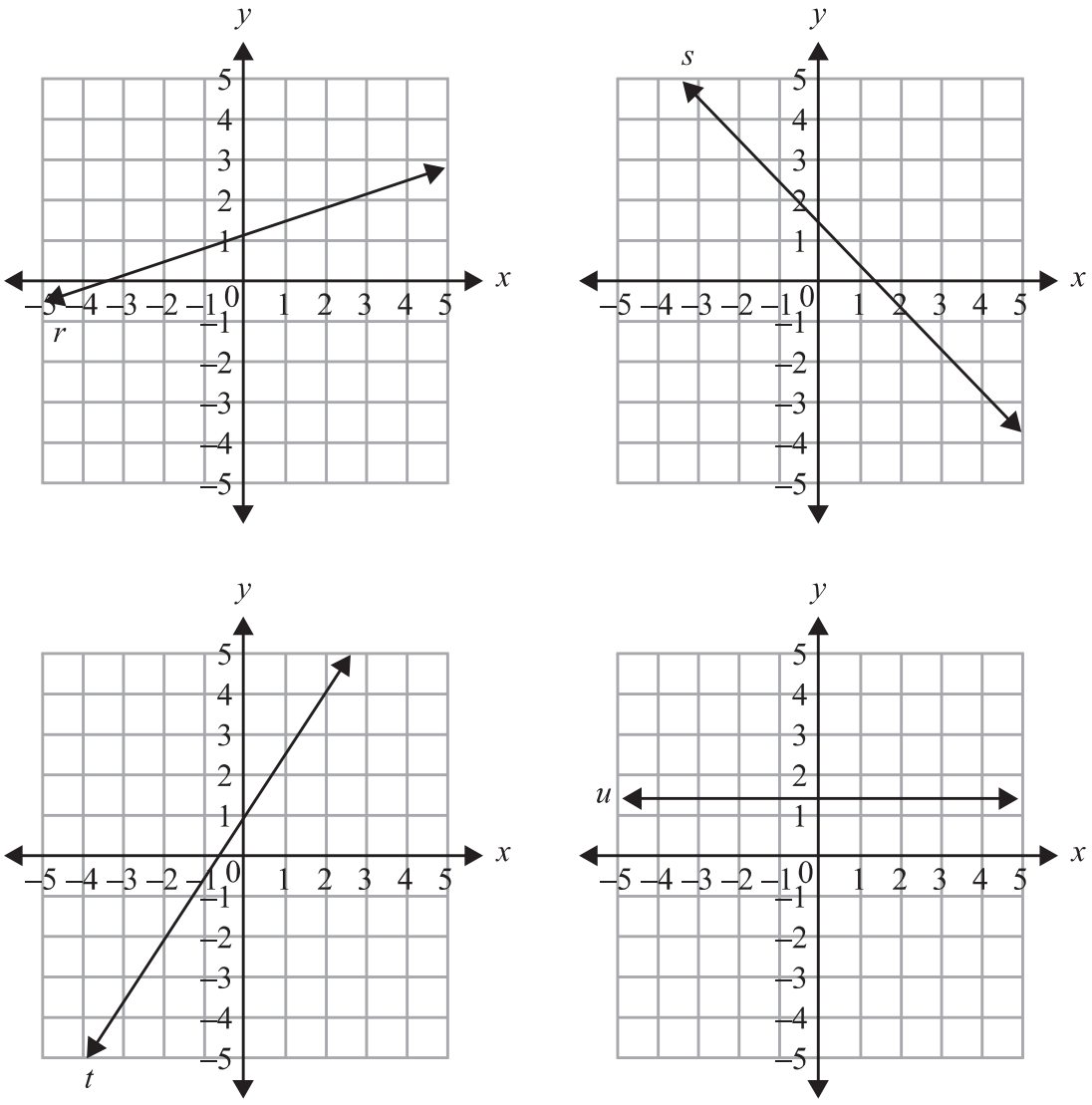
A. $3x^4 + 2x^3 - 2x^2 + x + 1$

B. $3x^4 + 2x^3 - 2x^2 - x + 9$

C. $3x^4 + 6x^3 - 2x^2 + x + 1$

D. $3x^4 + 6x^3 - 2x^2 - x + 1$

4. **Use the graphs below to answer the question.**



Which line shows a negative slope?

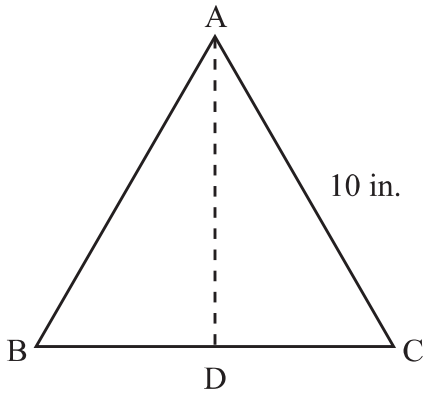
A. *r*

B. *s*

C. *t*

D. *u*

5. **Use the figure below to answer the question.**



△ ABC is equilateral and each side measures 10 inches (in.). Segment DC is equal to 5 in. What is the length of $\overline{AD}$?

A. $5\sqrt{2}$ in.

B. $5\sqrt{3}$ in.

C. $10\sqrt{2}$ in.

D. $10\sqrt{3}$ in.

6. What is the simplest form of the expression $\frac{2x^2+16x+14}{x^2-1}$?

A. $\frac{2(x+7)}{(x-1)}$

B. $\frac{(2x+2)(x+7)}{(x+1)(x-1)}$

C. $\frac{2(x+7)}{(x+1)}$

D. $\frac{2(x+1)(x+7)}{(x-1)(x-1)}$

7. What is the value of $\left(\frac{1}{2}\right)^{-2}$?

   A. -4

   B. $-\frac{1}{4}$

   C. $\frac{1}{4}$

   D. 4

8. A line has a slope of $\frac{3}{4}$ and passes through the point $(-4, 2)$. Which represents the equation of the line written in standard form?

   A. $4x - 3y = -22$

   B. $-3x + 4y = 20$

   C. $4x - 3y = -18$

   D. $3x - 4y = -24$

9. **Use the table below to answer the question.**

School Population By Gender

	Males	Females	Total
Freshman	8	17	25
Sophomores	15	10	25
Juniors	12	13	25
Seniors	15	10	25
Total	50	50	100

In a school of 100 students, what is the probability of randomly selecting a student that is either a female or a senior?

A. $\frac{10}{100}$

B. $\frac{25}{100}$

C. $\frac{60}{100}$

D. $\frac{65}{100}$

10. The vertices of $\triangle$ ABC are A (0, 0), B (6, 0), and C (3, 7). What type of triangle is $\triangle$ ABC?

A. acute isosceles triangle

B. acute scalene triangle

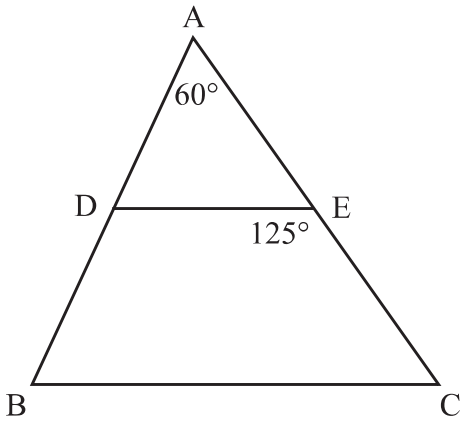
C. right isosceles triangle

D. right scalene triangle

11. Which statement is true for a line that passes through points (–2, 4) and (3, 4)?

    A. The line has a negative slope.
    B. The line has a positive slope.
    C. The line has an undefined slope.
    D. The line has a zero slope.

12. **Use the figure below to answer the question.**



In $\triangle$ ABC, $\overline{DE}$ is parallel to $\overline{BC}$. The measure of $\angle A$ is 60° and the measure of $\angle DEC$ is 125°. What is the measure of $\angle B$?

    A. 55°
    B. 60°
    C. 65°
    D. 70°

13. What is the value of the expression $\dfrac{3+\dfrac{1}{4}}{3}$?

A. $\dfrac{1}{4}$

B. $\dfrac{13}{12}$

C. $\dfrac{5}{4}$

D. $\dfrac{13}{3}$

14. Using positive exponents, what is the simplest form of the expression $\left(\dfrac{x^4}{3y^2}\right)^3$?

A. $\dfrac{x^7}{3y^2}$

B. $\dfrac{x^7}{27y^5}$

C. $\dfrac{x^{12}}{9y^6}$

D. $\dfrac{x^{12}}{27y^6}$

15. Given the linear equation $y = \frac{2}{3}x + 2$, which table of values represents this equation?

A.

x	*y*
−3	0
0	2
3	4

B.

x	*y*
0	2
2	5
3	4

C.

x	*y*
−3	0
2	0
6	6

D.

x	*y*
−9	−6
−6	−2
0	2

16. What is the *y*-intercept of the line that contains the points (3, 3) and (6, − 1)?

A. 5

B. 6

C. 7

D. 8

17. The coordinates of the vertices for rectangle ABCD are A (2, 4), B (6, 10), C (9, 8) and D (5, 2). What is the length of a diagonal ($\overline{AC}$ or $\overline{BD}$) of the rectangle?

A. $\sqrt{13}$

B. $\sqrt{52}$

C. $\sqrt{60}$

D. $\sqrt{65}$

18. **Use the frequency table below to answer the question.**

Frequency Table

Interval	Frequency
70-74	||
75-79	|
80-84	|||
85-89	|
90-94	~~||||~~ |
95-99	|

The frequency table shows the test scores for each interval. What effect would dropping the two scores in the 70-74 interval have on the mean and standard deviation?

A. both the mean and standard deviation increase

B. both the mean and standard deviation decrease

C. the mean increases, but the standard deviation decreases

D. the mean decreases, but the standard deviation increases

19. A car used 12 gallons of gasoline and traveled a total distance of 290 miles. The car's fuel efficiency is 25 miles per gallon on the highway and 20 miles per gallon in the city. The variable h will represent the number of gallons used on the highway. Which equation could be used to find h?

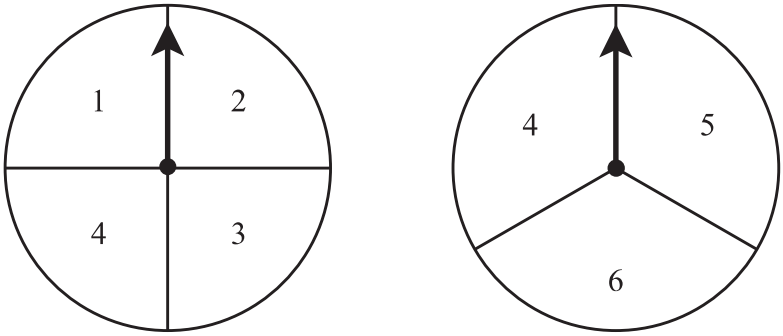
    A. $25h + 20(12 - h) = 290$

    B. $25(12 - h) + 20h = 290$

    C. $25(12 + h) + 20h = 290$

    D. $25h + 20(12 + h) = 290$

20. **Use the spinners below to answer the question.**



Spinner A Spinner B

All four sections on spinner A are the same size, and all three sections on spinner B are the same size. Charlie spins Spinner A and Spinner B one time each and adds his results. What is the probability of getting a sum of 6?

A. $\frac{1}{12}$

B. $\frac{2}{12}$

C. $\frac{3}{12}$

D. $\frac{4}{12}$

21. An item sells for $20 per kilogram. What is the cost in cents per gram for this item?

    A. 0.02 cents per gram

    B. 0.2 cents per gram

    C. 2 cents per gram

    D. 20 cents per gram

22. Nine tiles numbered 1 through 9 are placed in a bag. A tile is randomly drawn and replaced. Then a second tile is randomly drawn. What is the probability the first tile drawn and the second tile drawn are both even numbers?

    A. $\frac{8}{81}$

    B. $\frac{16}{81}$

    C. $\frac{20}{81}$

    D. $\frac{25}{81}$

23. John found the length of one side of a square to be $\sqrt{40}$ inches. Which is closest to the length of the side of the square?

    A. about 6 inches

    B. about 7 inches

    C. about 10 inches

    D. about 20 inches

24. Line l and line k are perpendicular. Line l has a slope of 3. Line k contains the points (5, 8) and (2, y). What is the value of y?

    A. –1

    B. 7

    C. 9

    D. 17

NeSA-M High School Reference Sheet

Shape	Area	Circumference
Circle	$A = \pi r^2$	$C = \pi d = 2\pi r$
Triangle	$A = \frac{1}{2} bh$	**Perimeter**
Rectangle	$A = lw$	$P = 2l + 2w$
Trapezoid	$A = \frac{1}{2} h\,(b_1 + b_2)$	
Parallelogram	$A = bh$	

Key	
b = base	w = width
B = area of base	d = diameter
h = height	r = radius
l = length	ℓ = slant
Use 3.14 for π	

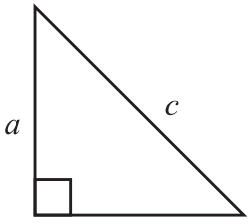
3-Dimensional Shape	Volume	Total Surface Area
Right Circular Cone	$V = \frac{1}{3} \pi r^2 h$	$T = \frac{1}{2}(2\pi r)\ell + \pi r^2 = \pi r\ell + \pi r^2$
Pyramid	$V = \frac{1}{3} Bh$	$T = B + \frac{1}{2} P\ell$
Sphere	$V = \frac{4}{3} \pi r^3$	$T = 4\pi r^2$
Right Circular Cylinder	$V = \pi r^2 h$	$T = 2\pi rh + 2\pi r^2$
Right Prism	$V = Bh$	$T = 2B + Ph$

Formulas

Distance, rate, and time formula, where d = distance, r = rate, t = time:

$$d = rt$$

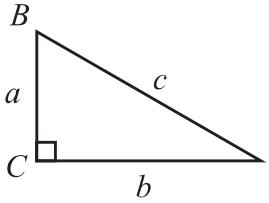
Pythagorean Theorem



$$c^2 = a^2 + b^2$$

Right-Triangle Relationships

Trigonometric Ratios

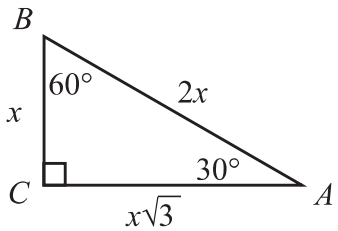


$$\sin A = \frac{a}{c}$$

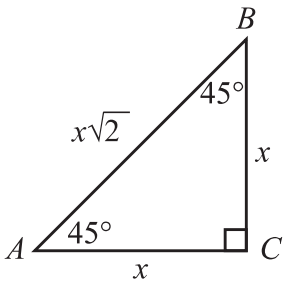
$$\cos A = \frac{b}{c}$$

$$\tan A = \frac{a}{b}$$

30°-60°-90° Triangle Relationships



45°-45°-90° Triangle Relationships



NeSA-M High School Reference Sheet

Linear Equation Forms

Point-Slope Form:

$$y - y_1 = m\,(x - x_1)$$

Standard or General Form:

$$Ax + By = C$$

Slope-Intercept Form:

$$y = mx + b$$

Coordinate Geometry

Given: Points $A(x_1, y_1)$, $B(x_2, y_2)$

Distance between two points:

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Midpoint between two points:

$$\text{Midpoint of } \overline{AB} = \left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}\right)$$

Slope of line through two points: $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

Equation of a Circle

$$(x - h)^2 + (y - k)^2 = r^2$$

(h, k) = center $\qquad$ r = radius

Quadratic Formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$ax^2 + bx + c = 0$$

Standard Units	Metric Units
Conversions – Length	
1 foot (ft) = 12 inches (in.)	1 centimeter (cm) = 10 millimeters (mm)
1 yard (yd) = 3 feet (ft) = 36 inches (in.)	1 meter (m) = 100 centimeters (cm)
1 mile (mi) = 1,760 yards (yd) = 5,280 feet (ft)	1 kilometer (km) = 1,000 meters (m)
Conversions – Area	
1 square foot (sq. ft) = 144 square inches (sq. in.)	
1 square yard (sq. yd) = 9 square feet (sq. ft)	
Conversions – Volume	
1 cubic yard (cu. yd) = 27 cubic feet (cu. ft)	
1 cubic foot (cu. ft) = 1,728 cubic in. (cu. in.)	
Conversions – Capacity	
1 cup = 8 fluid ounces (fl oz)	1 liter (l) = 1,000 milliliters (ml)
1 pint (pt) = 2 cups	1 liter (l) = 1,000 cubic centimeters (cu. cm)
1 quart (qt) = 2 pints (pt)	1 kiloliter (kl) = 1,000 liters (l)
1 gallon (gal.) = 4 quarts (qt)	
Conversions – Weight/Mass	
1 pound (lb) = 16 ounces (oz)	1 gram (g) = 1,000 milligrams (mg)
1 ton = 2,000 pounds (lb)	1 kilogram (kg) = 1,000 grams (g)

Grade 11 Practice Test
Answer Key

Number	Key
1	C
2	B
3	D
4	B
5	B
6	A
7	D
8	B
9	D
10	A
11	D
12	C
13	B
14	D
15	A
16	C
17	D
18	C
19	A
20	B
21	C
22	B
23	A
24	C

THE 44th ANNUAL (2023) UNIVERSITY OF MARYLAND

HIGH SCHOOL MATHEMATICS COMPETITION

PART I MULTIPLE CHOICE

For each of the following 25 questions, carefully blacken the appropriate box on the answer sheet with a #2 pencil. Do not fold, bend or write stray marks on either side of the answer sheet. Each correct answer is worth 6 points. Two points are awarded for answers left blank. Zero points are given for incorrect answers. Note that wild guessing is apt to lower your score. When the exam is over, give your answer sheet to the proctor. You may keep your copy of the questions.

NO CALCULATORS

75 MINUTES

1. An ant walks a distance $A = 10^9$ millimeters. A bear walks $B = 10^6$ feet. A chicken walks $C = 10^8$ inches. What is the correct ordering of A, B, C? (Note: There are 25.4 millimeters in an inch, and there are 12 inches in a foot.)

   a. $A < B < C$ b. $A < C < B$ c. $C < B < A$ d. $B < A < C$ e. $B < C < A$

2. Peter Rabbit is hopping along the number line, always jumping in the positive x direction. For his first jump, he starts at 0 and jumps 1 unit to get to the number 1. For his second jump, he jumps 4 units to get to the number 5. He continues jumping by jumping 1 unit whenever he is on a multiple of 3 and by jumping 4 units whenever he is on a number that is not a multiple of 3. What number does he land on at the end of his 100th jump?

   a. 297 b. 298 c. 301 d. 302 e. 303

3. Adam is walking in the city. In order to get around a large building, he walks 12 miles east and then 5 miles north, then stops. His friend Neutrino, who can go through buildings, starts in the same place as Adam but walks in a straight line to where Adam stops. How much farther than Neutrino does Adam walk?

   a. 1 mile b. 2 miles c. 3 miles d. 4 miles e. 5 miles

4. Euler is selling Mathematician cards to Gauss. Three Fermat cards plus 5 Newton cards costs 95 Euros, while 5 Fermat cards plus 2 Newton cards also costs 95 Euros. How many Euros does one Fermat card cost?

   a. 10 b. 15 c. 20 d. 30 e. 35

5. You shoot an arrow in the air. It falls to earth, you know not where. But you do know that the arrow's height in feet after t seconds is $-16t^2 + 80t + 96$. After how many seconds does the arrow hit the ground? (the ground has height 0)

   a. 2 b. 3 c. 4 d. 5 e. 6

6. Let
$$A = \log(1) + \log(2) + \log(3) + \cdots + \log(2023)$$
and let
$$B = \log(1/1) + \log(1/2) + \log(1/3) + \cdots + \log(1/2023).$$
(logs are logs base 10)

What is the value of $A + B$?

a. 0 b. 1 c. $-\log(2023!)$ d. $\log(2023!)$ e. -2023

7. Suppose $S = \{1, 2, 3, x\}$ is a set with four distinct real numbers for which the difference between the largest and smallest values of S is equal to the sum of elements of S. What is the value of x?

a. -1 b. -3/2 c. -2 d. -2/3 e. -3

8. How many positive integers less than 1 million have exactly 5 positive divisors?

a. 1 b. 5 c. 11 d. 23 e. 24

9. The Amazing Prime company ships its products in boxes whose length, width, and height (in inches) are prime numbers. If the volume of one of their boxes is 105 cubic inches, what is its surface area (that is, the sum of the areas of the 6 sides of the box) in square inches?

a. 21 b. 71 c. 77 d. 105 e. 142

10. There are 100 people in a room. Some are *wise* and some are *optimists*.

- A *wise* person can look at someone and know if they are wise or if they are an optimist.
- An *optimist* thinks everyone is wise (including themselves).

Everyone in the room writes down what they think is the number of wise people in the room. What is the smallest possible value for the average?

a. 10 b. 25 c. 50 d. 75 e. 100

11. Let S_1 be a square with side s and C_1 be the circle inscribed in it. Let C_2 be a circle with radius r and S_2 be a square inscribed in it. We are told that the area of $S_1 - C_1$ is the same as the area of $C_2 - S_2$. Which of the following numbers is closest to s/r?

a. 1 b. 2 c. 3 d. 4 e. 5

12. Suppose for real numbers a, b, c we know $a + \dfrac{1}{b} = 3$, and $b + \dfrac{3}{c} = \dfrac{1}{3}$. What is the value of $c + \dfrac{27}{a}$?

a. 1 b. 3 c. 8 d. 9 e. 21

13. The orthocenter of triangle ABC lies on its circumcircle. One of the angles of ABC must equal: (Note: The orthocenter of a triangle is the point where all three altitudes intersect.)

a. 30° b. 60° c. 90° d. 120° e. It cannot be deduced from the given information.

14. Let $m \neq -1$ be a real number. Consider the quadratic equation

$$(m+1)x^2 + 4mx + m - 3 = 0.$$

Which of the following must be true?

(I) Both roots of this equation must be real.

(II) If both roots are real, then one of the roots must be less than -1.

(III) If both roots are real, then one of the roots must be larger than 1.

a. Only (I) b. (I) and (II) c. Only (III) d. Both (I) and (III) e. (I), (II), and (III)

15. What is the least positive integer m such that the following is true?

Given m integers between 1 and 2023, inclusive, there must exist two of them a, b such that $1 < \frac{a}{b} \leq 2$.

a. 10 b. 11 c. 12 d. 13 e. 1415

16. How many integers between 123 and 789 have at least two identical digits, when written in base 10?

a. 180 b. 184 c. 186 d. 189 e. 191

17. The lengths of the sides of triangle $A'B'C'$ are equal to the lengths of the three medians of triangle ABC. Then the ratio $\text{Area}(A'B'C')/\text{Area}(ABC)$ equals

a. $\frac{1}{2}$ b. $\frac{2}{3}$ c. $\frac{3}{4}$ d. $\frac{5}{6}$ e. Cannot be determined from the information given.

18. How many ordered triples of integers (a, b, c) satisfy the following system?

$$\begin{cases} ab + c = 17 \\ a + bc = 19 \end{cases}$$

a. 2 b. 3 c. 4 d. 5 e. 6

19. Three positive real numbers a, b, c satisfy $a^b = 343, b^c = 10, a^c = 7$. Find b^b.

a. 1000 b. 900 c. 1200 d. 4000 e. 100

20. A strip is defined as the region between two parallel lines; the width of the strip is the distance between the two lines. Two strips of width 1 intersect in a parallelogram whose area is 2. What is the angle between the strips?

a. 15° b. 30° c. 45° d. 60° e. 90°

21. Let a, b, c, d, e be real numbers such that $a < b < c < d < e$. The least possible value of the function $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = |x-a| + |x-b| + |x-c| + |x-d| + |x-e|$ is

a. $e+d+c+b+a$ b. $e+d+c-b-a$ c. $e+d+|c|-b-a$ d. $e+d+b-a$ e. $e+d-b-a$

22. A sequence $a_1, a_2, \ldots$ satisfies $a_1 = \dfrac{5}{2}$ and $a_{n+1} = a_n^2 - 2$ for all $n \geq 1$. Let M be the integer which is closest to a_{2023}. The last digit of M equals

a. 0 b. 2 c. 4 d. 6 e. 8

23. Assume a triangle ABC satisfies $|AB| = 1, |AC| = 2$ and $\angle ABC = \angle ACB + 90^\circ$. What is the area of ABC?

a. 6/7 b. 5/7 c. 1/2 d. 4/5 e. 3/5

24. Bob is practicing addition in base 2. Each time he adds two numbers in base 2, he counts the number of carries. For example, when summing the numbers 1001 and 1011 in base 2,

$$\begin{array}{ccccccc} & {\scriptstyle 1} & & {\scriptstyle 1} & {\scriptstyle 1} & & \\ & 0 & 1 & 0 & 0 & 1 \\ & 0 & 1 & 0 & 1 & 1 \\ \hline & 1 & 0 & 1 & 0 & 0 \end{array}$$

there are three carries (shown on the top row). Suppose that Bob starts with the number 0, and adds 111 (i.e. 7 in base 2) to it one hundred times to obtain the number 1010111100 (i.e. 700 in base 2). How many carries occur (in total) in these one hundred calculations?

a. 280 b. 289 c. 291 d. 294 e. 297

25. Suppose that S is a set of real numbers between 2 and 8 inclusive, and that for any two elements $y > x$ in S,

$$98y - 102x - xy \;\geq\; 4.$$

What is the maximum possible size for the set S?

a. 12 b. 14 c. 16 d. 18 e. 20

44th ANNUAL UNIVERSITY OF MARYLAND
HIGH SCHOOL MATHEMATICS COMPETITION
PART II
November 29, 2023, 1:00–3:00 pm

Instructions: Solutions to different problems should go on separate pages. Write down your name and the problem number on the upper left corner of *all* pages that you submit for grading. Show your work and justify all of your steps. Submit only what you want to be graded; no blank paper or scratch paper.

NO CALCULATORS
2 hours

1. An Indian raga has two kinds of notes: a short note, which lasts for 1 beat, and a long note, which lasts for 2 beats. For example, there are 3 ragas which are 3 beats long: 3 short notes, a short note followed by a long note, and a long note followed by a short note. How many Indian ragas are 11 beats long? Justify your answer.

2. Let $n \geq 2$ be an integer. There are n houses in a town. All distances between pairs of houses are different. Every house sends a visitor to the house closest to it. Find all possible values of n (with full justification) for which we can design a town with n houses where every house is visited.

3. Let p be a prime, and $n > p$ be an integer. Prove that $\binom{n+p-1}{p} - \binom{n}{p}$ is divisible by n.

   Note: $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$.

4. Assume every side length of a triangle ABC is more than 2 and two of its angles are given by $\angle ABC = 57^\circ$ and $\angle ACB = 63^\circ$. Point P is chosen on side BC with $|BP| : |PC| = 2 : 1$. Points M, N are chosen on sides AB and AC, respectively so that $|BM| = 2$ and $|CN| = 1$. Let Q be the point on segment MN for which $|MQ| : |QN| = 2 : 1$. Find the value of $|PQ|$. Your answer must be in simplest form. Fully justify your answer.

5. Let $0 \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq 1$ be n real numbers with $n \geq 2$. Assume $a_1 + a_2 + \cdots + a_n \geq n - 1$. Prove that
$$a_2 a_3 \cdots a_n \geq \left(1 - \frac{1}{n}\right)^{n-1}.$$